Sharp Trace Hardy-Sobolev-Maz'ya Inequalities and the Fractional Laplacian

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Abstract

In this work we establish trace Hardy and trace Hardy-Sobolev-Maz'ya inequalities with best Hardy constants, for domains satisfying suitable geometric assumptions such as mean convexity or convexity. We then use them to produce fractional Hardy-Sobolev-Maz'ya inequalities with best Hardy constants for various fractional Laplacians. In the case where the domain is the half space our results cover the full range of the exponent $s \in (0,1)$ of the fractional Laplacians. We answer in particular an open problem raised by Frank and Seiringer [FS].

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Contents

1 Introduction and Main Results

The Hardy inequality in the upper half space asserts that

$$\int_{I\!R^n_+} |\nabla u|^2 dx \geq \frac{1}{4} \int_{I\!R^n_+} \frac{|u|^2}{x_n^2} dx, \qquad u \in C_0^\infty(I\!R^n_+), \tag{1.1}$$

where $I\!R^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$ denotes the upper half-space, and $\frac{1}{4}$ is the best possible constant.

If $\Omega \subset I\!R^n$ and $d(x) = \operatorname{dist}(x, \partial\Omega)$ then there are two main directions towards establishing Hardy inequalities. One direction is to find proper regularity assumptions on the boundary of Ω that imply the existence of a positive constant C_Ω such that

$$\int_\Omega |\nabla u|^2 dx \geq C_\Omega \int_\Omega \frac{|u|^2}{d^2(x)} dx \,, \qquad u \in C_0^\infty(\Omega) \,.$$

In this direction we refer to [A], [KK] and references therein.

A second direction aims at finding geometric assumptions on Ω that imply the Hardy inequality with best constant $\frac{1}{4}$, that is

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{d^2(x)} dx \,, \qquad u \in C_0^\infty(\Omega) \,. \tag{1.2}$$

The standard geometric assumption here is convexity of Ω, see, e.g., [D1], [D2], [BM]. However inequality (1.2) remains true under the weaker assumption

$$-\Delta d(x) \geq 0, \qquad x \in \Omega \,. \tag{1.3}$$

This is meant in the distributional sense. We refer to [BFT] where this condition arises in a natural way. In fact condition (1.3) is equivalent to convexity in two space dimensions, but it is weaker than convexity for $n \geq 3$, since any convex domain satisfies (1.3) whereas there are nonconvex domains that satisfy (1.3) [AK]. We emphasize that there is no need for further regularity assumptions on Ω. In case $\partial\Omega$ is C^2, condition (1.3) is recently shown to be equivalent to the mean convexity of $\partial\Omega$, that is $(n-1)H(x) = -\Delta d(x) \geq 0$ for $x \in \partial\Omega$, see [LLL], [P].

If in addition to (1.3) the domain Ω is a C^2 domain with finite inner radius then it has been established that one can combine the Sobolev and the Hardy inequality, the latter with best constant. More precisely, for $n \geq 3$ there exists a positive constant c such that

$$\int_\Omega |\nabla u|^2 dx \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{d^2(x)} dx + c \left(\int_\Omega |u|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}}, \quad u \in C_0^\infty(\Omega) \,, \tag{1.4}$$

see [FMT]. In [Gk] Hardy-Sobolev-Maz'ya inequalities are established under a different geometric assumption than (1.3), that allows infinite inner radius. Frank and Loss established in [FL] inequality (1.4) with a constant c independent of Ω, when Ω is convex.

Recently, a lot of attention is attracted by the fractional Laplacian. For $s \in (0,1)$ it is defined as follows

$$(-\Delta)^s f(x) = c_{n,s}\ P.V. \int_{I\!R^n} \frac{f(x)-f(\xi)}{|x-\xi|^{n+2s}} d\xi\ , \tag{1.5}$$

where P.V. stands for the Cauchy principal value and

$$c_{n,s} = \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)\pi^{\frac{n}{2}}}\ . \tag{1.6}$$

There are other ways for defining the fractional Laplacian, as for instance via the Fourier transform. We note that the fractional Laplacian is a non local operator and this raises several technical difficulties. However, there is a way of studying various properties of the fractional Laplacian via the Dirichlet to Neumann map. This has been recently studied by Caffarelli and Silvestre [CS], and it will be central in this work. Let us briefly recall the approach in [CS], where by adding a new variable y, they relate the fractional Laplacian to a local operator. For any function f one solves the following extension problem

$$div(y^{1-2s}\nabla_{(x,y)}u(x,y)) = 0, \quad I\!R^n \times (0,\infty), \tag{1.7}$$

$$u(x,0) = f(x), \quad I\!R^n\ , \tag{1.8}$$

the natural energy of which is given by

$$J[u] = \int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla_{(x,y)} u(x,y)|^2 dx dy.$$

Then, up to a normalizing factor C one establishes that

$$-\lim_{y\to 0^+} y^{1-2s} u_y(x,y) = C(-\Delta)^s f(x)\ .$$

Our interest in this work is to study the fractional Laplacian defined in subsets of $I\!R^n$ and in particular to establish Hardy and Hardy-Sobolev-Maz'ya inequalities there. There is a lot of interest in fractional Laplacian in subsets of $I\!R^n$ coming from various applications, as for instance censored stable processes and killed stable processes [CSo], [BBC], [CKS1], [CKS2], Gamma convergence and phase transition problems [ABS], [G], [SV1], [SV2], [PSV] and nonlinear PDE theory [CT], [T], [CC]. In [BD] it was conjectured that the best Hardy constant in the case of the fractional Laplacian associated to a censored stable process is the same for all convex domains. In [FS] it was posed the question establishing fractional Hardy-Sobolev-Maz'ya inequalities for the half space.

Contrary to the case of the full space $I\!R^n$, there are several different fractional Laplacians that one can define on a domain $\Omega \subsetneqq I\!R^n$. In particular in the above mentioned references three different fractional Laplacians appear. In all cases we will use the Dirichlet to Neumann map after identifying the proper extension problem. Throughout this work we assume that the domain Ω is a uniformly Lipschitz domain; for the precise definition see Section 2.

We start with the fractional Laplacian that appears in [CT], [T], [CC]. The proper extension problem in this case is to consider test functions in $C_0^\infty(\Omega \times I\!R)$. At this point we recall that the inner radius of a domain Ω is defined as $R_{in} := \sup_{x\in\Omega} d(x)$. We say that the domain Ω has finite inner radius whenever $R_{in} < \infty$. Our first result concerns the extended problem and reads:

Theorem 1.1. *(***Trace Hardy & Trace Hardy-Sobolev-Maz'ya I***)*
Let $\frac{1}{2} \le s < 1$, $n \ge 2$ and $\Omega \subsetneqq I\!R^n$ be a domain.
(i) *If in addition Ω is such that*

$$-\Delta d(x) \ge 0, \qquad x \in \Omega\ , \tag{1.9}$$

then for all $u \in C_0^\infty(\Omega \times I\!R)$ there holds

$$\int_0^{+\infty} \int_\Omega y^{1-2s} |\nabla_{(x,y)} u(x,y)|^2 dx dy \ge \bar{d}_s \int_\Omega \frac{u^2(x,0)}{d^{2s}(x)} dx\ , \tag{1.10}$$

with

$$\bar{d}_s := \frac{2\Gamma\left(1-s\right)\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)\Gamma\left(s\right)} \,. \tag{1.11}$$

(ii) *Suppose there exists a point $x_0 \in \partial\Omega$ and $r > 0$ such that the part of the boundary $\partial\Omega \cap B(x_0, r)$ is C^1 regular. Then*

$$\bar{d}_s \geq \inf_{u \in C_0^\infty(\Omega \times I\!R)} \frac{\int_0^{+\infty}\int_\Omega y^{1-2s}|\nabla u|^2 dxdy}{\int_\Omega \frac{u^2(x,0)}{d^{2s}(x)} dx} \,.$$

In particular $\bar{d}_s$ in (1.10) is the best constant.
(iii) *If Ω is a uniformly Lipschitz domain with finite inner radius satisfying (1.9), and $s \in (\frac{1}{2}, 1)$, then there exists a positive constant c such that for all $u \in C_0^\infty(\Omega \times I\!R)$ there holds*

$$\int_0^{+\infty}\int_\Omega y^{1-2s}|\nabla_{(x,y)}u(x,y)|^2 dxdy \geq \bar{d}_s \int_\Omega \frac{u^2(x,0)}{d^{2s}(x)} dx + c\left(\int_\Omega |u(x,0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}} \,. \tag{1.12}$$

Actually, in the case of half space $\Omega = I\!R^n_+$ we establish a much stronger result covering the full range $s \in (0,1)$. In particular we have

Theorem 1.2. *(***Half Space, Trace Hardy-Sobolev-Maz'ya I***)*
Let $0 < s < 1$ and $n \geq 2$.
(i) *For all $u \in C_0^\infty(I\!R^n_+ \times I\!R)$ there holds*

$$\int_0^{\infty}\int_{I\!R^n_+} y^{1-2s}|\nabla_{(x,y)}u(x,y)|^2 dxdy \geq \bar{d}_s \int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx \,, \tag{1.13}$$

with

$$\bar{d}_s := \frac{2\Gamma\left(1-s\right)\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)\Gamma\left(s\right)} \,. \tag{1.14}$$

(ii) *The constant $\bar{d}_s$ in (1.13) is sharp, that is*

$$\bar{d}_s = \inf_{u \in C_0^\infty(I\!R^n_+ \times I\!R)} \frac{\int_0^{\infty}\int_{I\!R^n_+} y^{1-2s}|\nabla u|^2 dxdy}{\int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx} \,.$$

(iii) *There exists a positive constant c such that for all $u \in C_0^\infty(I\!R^n_+ \times I\!R)$ there holds*

$$\int_0^{\infty}\int_{I\!R^n_+} y^{1-2s}|\nabla_{(x,y)}u(x,y)|^2 dxdy \geq \bar{d}_s \int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx + c\left(\int_{I\!R^n_+} |u(x,0)|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}} \,. \tag{1.15}$$

We will apply Theorem 1.1 to the fractional Laplacian that is defined as follows. Let $\Omega \subset I\!R^n$ be a bounded domain, and λ_i and ϕ_i be the Dirichlet eigenvalues and orthonormal eigenfunctions of the Laplacian, i.e. $-\Delta\phi_i = \lambda_i\phi_i$ in Ω, with $\phi_i = 0$ on $\partial\Omega$. Then, for $f(x) = \sum c_i\phi_i(x)$ we define

$$(-\Delta)^s f(x) = \sum_{i=1}^{\infty} c_i \lambda_i^s \phi_i(x), \qquad 0 < s < 1 \,, \tag{1.16}$$

in which case

$$((-\Delta)^s f, f)_\Omega = \int_\Omega f(x)\,(-\Delta)^s f(x) dx = \sum_{i=1}^{\infty} c_i^2 \lambda_i^s. \tag{1.17}$$

In the sequel we will refer to this fractional Laplacian as the *spectral fractional Laplacian*. We then have

Theorem 1.3. *(***Hardy & Hardy-Sobolev-Maz'ya for Spectral Fractional Laplacian***)*
Let $\frac{1}{2} \le s < 1$, $n \ge 2$ and $\Omega \subset I\!R^n$ be a bounded domain.
(i) *If in addition Ω is such that*

$$-\Delta d(x) \ge 0, \qquad\qquad x \in \Omega\ , \tag{1.18}$$

then, for all $f \in C_0^\infty(\Omega)$ there holds

$$((-\Delta)^s f, f)_\Omega \ge d_s \int_\Omega \frac{f^2(x)}{d^{2s}(x)} dx\ , \tag{1.19}$$

with

$$d_s := \frac{2^{2s}\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)}\ . \tag{1.20}$$

(ii) *Suppose there exists a point $x_0 \in \partial\Omega$ and $r > 0$ such that the part of the boundary $\partial\Omega \cap B(x_0, r)$ is C^1 regular. Then*

$$d_s \ge \inf_{f \in C_0^\infty(\Omega)} \frac{((-\Delta)^s f, f)_\Omega}{\int_\Omega \frac{f^2(x)}{d^{2s}(x)} dx}\ .$$

(iii) *If Ω is a Lipschitz domain satisfying (1.18) and $s \in (\frac{1}{2}, 1)$, then there exists a positive constant c such that for all $f \in C_0^\infty(\Omega)$ there holds*

$$((-\Delta)^s f, f)_\Omega \ge d_s \int_\Omega \frac{f^2(x)}{d^{2s}(x)} dx + c \left(\int_\Omega |f(x)|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}}\ . \tag{1.21}$$

We next consider the fractional Laplacian associated to the killed stable processes that appears in [BD], [BBC], [SV1], [SV2], [PSV], which from now on we will call it *Dirichlet fractional Laplacian*. The proper extension problem involves test functions $u \in C_0^\infty(I\!R^n \times I\!R)$ such that $u(x, 0) = 0$ in the complement of Ω, that is, for $x \in \mathcal{C}\Omega$. For this fractional Laplacian, our assumption on the domain Ω is convexity instead of (1.3). The reason for this is that our method requires subharmonicity of the distance function in $\mathcal{C}\Omega$ which is equivalent to the convexity of Ω, see [AK]. Our next result reads:

Theorem 1.4. *(***Trace Hardy & Trace Hardy-Sobolev-Maz'ya II***)*
Let $\frac{1}{2} \le s < 1$, $n \ge 2$ and $\Omega \subsetneqq I\!R^n$ be a domain.
(i) *If in addition Ω is convex then, for all $u \in C_0^\infty(I\!R^n \times I\!R)$ such that $u(x, 0) = 0$ for $x \in \mathcal{C}\Omega$, there holds*

$$\int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla_{(x,y)} u(x, y)|^2 dx dy \ge \bar{k}_s \int_\Omega \frac{u^2(x, 0)}{d^{2s}(x)} dx\ , \tag{1.22}$$

with

$$\bar{k}_s := \frac{2^{1-2s}\Gamma^2(s + \frac{1}{2})\Gamma(1 - s)}{\pi\Gamma(s)}\ . \tag{1.23}$$

(ii) *Suppose there exists a point $x_0 \in \partial\Omega$ and $r > 0$ such that the part of the boundary $\partial\Omega \cap B(x_0, r)$ is C^1 regular. Then*

$$\bar{k}_s \ge \inf_{\substack{u \in C_0^\infty(I\!R^n \times I\!R), \\ u(x,0) = 0,\ x \in \mathcal{C}\Omega}} \frac{\int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla u|^2 dx dy}{\int_\Omega \frac{u^2(x,0)}{d^{2s}(x)} dx}\ .$$

In particular $\bar{k}_s$ in (1.22) is the best constant.
(iii) *If Ω is a uniformly Lipschitz and convex domain with finite inner radius and $s \in (\frac{1}{2}, 1)$, then there exists a positive constant c, such that the following improvement holds true for all $u \in C_0^\infty(I\!R^n \times I\!R)$ with $u(x, 0) = 0$ for $x \in \mathcal{C}\Omega$:*

$$\int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla_{(x,y)} u(x, y)|^2 dx dy \ge \bar{k}_s \int_\Omega \frac{u^2(x, 0)}{d^{2s}(x)} dx + c \left(\int_\Omega |u(x, 0)|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}}\ , \tag{1.24}$$

Elementary manipulations show that

$$\bar{d}_s = 2\sin^2\left(\frac{(2s+1)\pi}{4}\right)\ \bar{k}_s\ ,$$

thus

$$\bar{d}_s > \bar{k}_s\ , \qquad \text{for} \quad s \in (0,1)\ ,$$

which implies in particular that the best constants of Theorems 1.1 and 1.4 are different.

We next apply Theorem 1.4 to the Dirichlet fractional Laplacian. In this case, for $f \in C_0^\infty(\Omega)$ we extend f in all of $I\!R^n$ by setting $f = 0$ in $\mathcal{C}\Omega$ and use (1.5). In particular, the corresponding quadratic form is

$$\begin{aligned} ((-\Delta)_D^s f, f)_{I\!R^n} &= \frac{c_{n,s}}{2}\int_{I\!R^n}\int_{I\!R^n}\frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2s}}dxd\xi \\ &= \frac{c_{n,s}}{2}\left(\int_\Omega\int_\Omega\frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2s}}dxd\xi + 2\int_\Omega\int_{\mathcal{C}\Omega}\frac{|f(x)|^2}{|x-\xi|^{n+2s}}dxd\xi\right)\ , \end{aligned} \tag{1.25}$$

with the constant $c_{n,s}$ as given by (1.6). We then have:

Theorem 1.5. *(***Hardy & Hardy-Sobolev-Maz'ya for the Dirichlet Fractional Laplacian***)*
Let $\frac{1}{2} \le s < 1$, $n \ge 2$ and $\Omega \subsetneqq I\!R^n$ be a domain.
(i)*If in addition Ω is convex, then for all $f \in C_0^\infty(\Omega)$ there holds*

$$((-\Delta)_D^s f, f)_{I\!R^n} \ge \frac{\Gamma^2\left(s+\frac{1}{2}\right)}{\pi}\int_\Omega \frac{f^2(x)}{d^{2s}(x)}dx\ . \tag{1.26}$$

Equivalently, one has that

$$\int_{I\!R^n}\int_{I\!R^n}\frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2s}}dxd\xi \ge k_{n,s}\int_\Omega\frac{f^2(x)}{d^{2s}(x)}dx\ , \tag{1.27}$$

where

$$k_{n,s} := \frac{2^{1-2s}\pi^{\frac{n-2}{2}}\Gamma(1-s)\Gamma^2(s+\frac{1}{2})}{s\Gamma(\frac{n+2s}{2})}\ . \tag{1.28}$$

(ii) *Suppose there exists a point $x_0 \in \partial\Omega$ and $r > 0$ such that the part of the boundary $\partial\Omega \cap B(x_0, r)$ is C^1 regular. Then the Hardy constants $\frac{\Gamma^2\left(s+\frac{1}{2}\right)}{\pi}$ in (1.26) and $k_{n,s}$ in (1.27) are optimal.*
(iii) *If Ω is a uniformly Lipschitz and convex domain with finite inner radius and $s \in (\frac{1}{2}, 1)$, then there exists a positive constant c such that for all $f \in C_0^\infty(\Omega)$ there holds*

$$((-\Delta)_D^s f, f)_{I\!R^n} \ge \frac{\Gamma^2\left(s+\frac{1}{2}\right)}{\pi}\int_\Omega\frac{f^2(x)}{d^{2s}(x)}dx + c\left(\int_\Omega |f(x)|^{\frac{2n}{n-2s}}dx\right)^{\frac{n-2s}{n}}\ . \tag{1.29}$$

Equivalently, one has that

$$\int_{I\!R^n}\int_{I\!R^n}\frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2s}}dxd\xi \ge k_{n,s}\int_\Omega\frac{f^2(x)}{d^{2s}(x)}dx + c\left(\int_\Omega |f(x)|^{\frac{2n}{n-2s}}dx\right)^{\frac{n-2s}{n}}\ . \tag{1.30}$$

The case where Ω is the half–space $\Omega = I\!R^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$ is of particular interest see [BD], [BBC], [FS], [D], [S]. In this case we obtain a stronger result that covers the full range $s \in (0,1)$. More precisely we have:

Theorem 1.6. *(***Half Space, Trace Hardy-Sobolev-Maz'ya & Fractional Hardy-Sobolev-Maz'ya II***)* *Let $0 < s < 1$ and $n \geq 2$.*
(i) *Then for all $u \in C_0^\infty(I\!R^n \times I\!R)$ with $u(x,0) = 0$, $x \in I\!R^n_-$, there holds*

$$\int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla_{(x,y)} u(x,y)|^2 dx dy \geq \bar{k}_s \int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx \, , \tag{1.31}$$

where

$$\bar{k}_s := \frac{2^{1-2s} \Gamma^2(s+\frac{1}{2}) \Gamma(1-s)}{\pi \Gamma(s)} \, ,$$

is the best constant in (1.31).
(ii) *There exists a positive constant c, such that for all $u \in C_0^\infty(I\!R^n \times I\!R)$ with $u(x,0) = 0$, $x \in I\!R^n_-$, there holds*

$$\int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla_{(x,y)} u(x,y)|^2 dx dy \geq \bar{k}_s \int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx + c \left(\int_{I\!R^n_+} |u(x,0)|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}} \, , \tag{1.32}$$

(iii) *As a consequence, there exists a positive constant c such that for all $f \in C_0^\infty(I\!R^n_+)$ there holds*

$$\int_{I\!R^n} \int_{I\!R^n} \frac{|f(x) - f(\xi)|^2}{|x-\xi|^{n+2s}} dx d\xi \geq k_{n,s} \int_{I\!R^n_+} \frac{f^2(x)}{x_n^{2s}} dx + c \left(\int_{I\!R^n_+} |f(x)|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}} \, , \tag{1.33}$$

where $k_{n,s}$ is given by (1.28).

Or, equivalently, for all $f \in C_0^\infty(I\!R^n_+)$ there holds

$$\int_{I\!R^n_+} \int_{I\!R^n_+} \frac{|f(x) - f(\xi)|^2}{|x-\xi|^{n+2s}} dx d\xi \geq \kappa_{n,s} \int_{I\!R^n_+} \frac{f^2(x)}{x_n^{2s}} dx + c \left(\int_{I\!R^n_+} |f(x)|^{\frac{2n}{n-2s}} dx \right)^{\frac{n-2s}{n}} \, , \tag{1.34}$$

where

$$\kappa_{n,s} := \pi^{\frac{n-1}{2}} \frac{\Gamma(s+\frac{1}{2})}{s \Gamma(\frac{n+2s}{2})} \left[\frac{2^{1-2s}}{\sqrt{\pi}} \Gamma(1-s) \Gamma(s+\frac{1}{2}) - 1 \right] \, .$$

We note that the Hardy–Sobolev–Maz'ya inequality (1.33) refers to the Dirichlet fractional Laplacian, associated to the killed stable processes whereas inequality (1.34) is associated to the censored stable processes. The Hardy constants $k_{n,s}$ and $\kappa_{n,s}$ appearing in (1.33) and (1.34) respectively are optimal, as shown in [BD]. The corresponding fractional Hardy inequality of (1.34) with best constant, in the case of a convex domain Ω, that is,

$$\int_\Omega \int_\Omega \frac{|f(x) - f(\xi)|^2}{|x-\xi|^{n+2s}} dx d\xi \geq \kappa_{n,s} \int_\Omega \frac{f^2(x)}{d(x)^{2s}} dx, \qquad f \in C_0^\infty(\Omega) \, ,$$

has been established for $s \in (\frac{1}{2}, 1)$ in [LS]. The question of obtaining a Hardy–Sobolev–Maz'ya inequality for the half space was raised in [FS] and was answered positively in [S], [D], but only for the range $s \in (\frac{1}{2}, 1)$.

For other type of trace Hardy inequalities we refer to [DDM] and [AFV]. We finally note that fractional Sobolev inequalities play an important role in many other directions, see e.g., [BBM], [CG], [MS], [N].

2 The Trace Hardy inequality I

In this section we will prove the trace Hardy inequality contained in Theorem 1.1. We first recall the definition of a uniformly Lipschitz domain Ω; see section 12 of [L]. We note that Stein calls such a domain minimally smooth, see section 3.3 of [St].

A domain Ω is called uniformly Lipschitz if there exist $\varepsilon > 0$, $L > 0$, and $M \in I\!N$ and a locally finite countable cover $\{U_i\}$ of $\partial\Omega$ with the following properties:
(i) If $x \in \partial\Omega$ then $B(x, \varepsilon) \subset U_i$ for some i.
(ii) Every point of $I\!R^n$ is contained in at most M U_i's.
(iii) For each i there exist local coordinates $y = (y', y_n) \in I\!R^{n-1} \times I\!R$ and a Lipschitz function $f : I\!R^{n-1} \to I\!R$, with $Lipf \leq L$ such that

$$U_i \cap \Omega = U_i \cap \{(y', y_n) \in I\!R^{n-1} \times I\!R : y_n > f(y')\}.$$

Under the uniformly Lipschitz assumption on Ω the extension operator is defined in $W^{1,p}(\Omega)$, for all $p \geq 1$. We also note that when Ω is a bounded domain the above definition reduces to Ω being Lipschitz.

In the sequel we set $a = 1 - 2s$. Since $0 < s < 1$ we also have $-1 < a < 1$. We first establish the following useful identity:

Lemma 2.1. *Suppose that $a \in (-1, 1)$ and let $u \in C_0^\infty(\Omega \times I\!R)$ and $\phi \in C^2(\Omega \times (0, \infty)) \cap C(\bar{\Omega} \times [0, \infty))$ is such that $\phi(x, y) > 0$ in $\Omega \times [0, \infty)$, $\phi(x, y) = 0$ in $\partial\Omega \times (0, \infty)$,*

$$|y^a \frac{\phi_y(x,y)}{\phi(x,y)}| \leq V(x), \quad y \in (0,1), \quad x \in \Omega, \qquad 0 \leq V(x) \in L^1_{loc}(\Omega),$$

and for a.e. $x \in \Omega$, the following limit exists:

$$\lim_{y \to 0^+} \left(y^a \frac{\phi_y(x,y)}{\phi(x,y)} \right) .$$

We also require that the following integrals are finite

$$\int_0^{+\infty} \int_\Omega y^a \frac{|\nabla\phi|^2}{\phi^2} u^2 dxdy, \qquad \int_0^{+\infty} \int_\Omega \frac{|\mathrm{div}(y^a \nabla\phi)|}{\phi} u^2 dxdy .$$

We then have the identity:

$$\begin{aligned} \int_0^{+\infty} \int_\Omega y^a |\nabla u|^2 dxdy = & - \int_\Omega \lim_{y \to 0^+} \left(y^a \frac{\phi_y}{\phi} \right) u^2(x,0) dx + \int_0^{+\infty} \int_\Omega y^a |\nabla u - \frac{\nabla\phi}{\phi} u|^2 dxdy \\ & - \int_0^{+\infty} \int_\Omega \frac{\mathrm{div}(y^a \nabla\phi)}{\phi} u^2 dxdy. \end{aligned} \tag{2.1}$$

Proof: Expanding the square and integrating by parts we compute for $\varepsilon > 0$,

$$\begin{aligned} \int_\varepsilon^\infty \int_\Omega y^a |\nabla u - \frac{\nabla\phi}{\phi} u|^2 dxdy &= \int_\varepsilon^\infty \int_\Omega y^a \left(|\nabla u|^2 + \frac{|\nabla\phi|^2}{\phi^2} u^2 - \frac{\nabla\phi}{\phi} \nabla u^2 \right) dxdy \\ = \int_\varepsilon^\infty \int_\Omega y^a |\nabla u|^2 dxdy &+ \int_\varepsilon^\infty \int_\Omega \frac{\mathrm{div}(y^a \nabla\phi)}{\phi} u^2 dxdy + \int_\Omega \varepsilon^a \frac{\phi_y(x,\varepsilon)}{\phi(x,\varepsilon)} u^2(x,\varepsilon) dx . \end{aligned}$$

We then pass to limit $\varepsilon \to 0$ and the result follows easily.

□

We will use Lemma 2.1 with the following choice: $\phi(x,y) = d^{-\frac{a}{2}}(x) A\left(\frac{y}{d(x)}\right)$ for $y > 0$, $x \in \Omega$. The function A solves the following boundary value problem

$$(t^3 + t)A'' + (a + t^2(2+a))A' + \frac{(2+a)a}{4} tA = 0, \quad t > 0, \tag{2.2}$$

with

$$A(0) = 1, \qquad \lim_{t\to+\infty} A(t) = 0\,. \tag{2.3}$$

Equation (2.2) can also be written in divergence form as

$$(t^a(1+t^2)A')' + \frac{(2+a)a}{4} t^a A = 0. \tag{2.4}$$

From now on we will use the following notation:

$$f \sim g, \qquad in \;\; U,$$

whenever there exist positive constants c_1, c_2, such that

$$c_1 g \le f \le c_2 g, \qquad in \;\; U\,.$$

We then have the following

Proposition 2.2. *Suppose that $a \in (-1,1)$. The boundary value problem (2.2), (2.3) has a positive decreasing solution A with the following properties:*
(i) There exists a positive constant $\bar{d}_s$ such that

$$\lim_{t\to 0^+} t^a A'(t) = -\bar{d}_s\,,$$

with

$$\bar{d}_s = \frac{(1-a)\Gamma\left(\frac{1+a}{2}\right)\Gamma^2\left(\frac{4-a}{4}\right)}{\Gamma^2\left(\frac{2+a}{4}\right)\Gamma\left(\frac{3-a}{2}\right)} = \frac{2s\Gamma\left(1-s\right)\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)\Gamma\left(1+s\right)}.$$

(ii) For all $t>0$,

$$\begin{aligned} A(t) &\sim (1+t^2)^{-\frac{2+a}{4}}\,, \\ A'(t) &\sim -t^{-a}(1+t^2)^{-\frac{4-a}{4}}\,. \end{aligned}$$

Moreover,

$$\lim_{t\to+\infty} \frac{tA'(t)}{A(t)} = -\frac{2+a}{2}\,.$$

(iii) There holds:

$$\bar{d}_s = \int_0^\infty t^a(1+t^2)(A')^2 dt - \frac{(2+a)a}{4}\int_0^\infty t^a A^2 dt, \tag{2.5}$$

(iv) In case $a \in (-1,0]$, we have

$$tA'(t) + \frac{a}{2}A(t) \le 0\,.$$

Moreover for $a\in(-1,0)$ and all $t>0$ we have

$$tA'(t) + \frac{a}{2}A(t) \sim -A(t)\,.$$

Proof: We change variables in (2.2) by $z = -t^2$ and define $B(z)$ such that $A(t) = B(-t^2)$, whence $A_t = -2tB_z$ and $A_{tt} = -2B_z + 4t^2 B_{zz}$. It then follows that $B(z)$ satisfies the Gauss hypergeometric equation

$$z(1-z)B'' + \left(\frac{1+a}{2} - \frac{3+a}{2}z\right)B' - \frac{a(2+a)}{16}B = 0, \qquad -\infty < z < 0,$$

whose general solution is given by

$$B(z) = C_1 F_1\left(\frac{a}{4}, \frac{2+a}{4}, \frac{1+a}{2}; z\right) + C_2 z^{\frac{1-a}{2}} F_2\left(\frac{2-a}{4}, \frac{4-a}{4}, \frac{3-a}{2}; z\right)\;;$$

see [AS], Section 15.5 as well as 15.1 for the definition and basic properties of the function F. It follows that

$$A(t)=C_1F_1\left(\frac{a}{4},\frac{2+a}{4},\frac{1+a}{2};-t^2\right)+C_2t^{1-a}e^{\frac{i\pi(1-a)}{2}}F_2\left(\frac{2-a}{4},\frac{4-a}{4},\frac{3-a}{2};-t^2\right). \tag{2.6}$$

Since $F(\alpha,\beta,\gamma;0)=1$ for any α, β, γ, the condition $A(0)=1$ implies that $C_1=1$. We then have

$$\begin{aligned}\bar{d}_s &= -\lim_{t\to 0^+} t^aA'(t)\\ &= -\lim_{t\to 0^+} t^a(-2tF_1'+(1-a)C_2e^{\frac{i\pi(1-a)}{2}}t^{-a}F_2-2C_2t^{2-a}e^{\frac{i\pi(1-a)}{2}}F_2')\\ &= -(1-a)C_2e^{\frac{i\pi(1-a)}{2}}.\end{aligned} \tag{2.7}$$

In the above calculation we have also used the fact that

$$F'(\alpha,\beta,\gamma;z)=\frac{d}{dz}F(\alpha,\beta,\gamma;z)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z).$$

We next compute the behavior of A at infinity. To this end we will use the inversion formula, valid for any α, β, γ and $|arg(-z)|<\pi$:

$$\begin{aligned}F(\alpha,\beta,\gamma;z)= &\quad \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha,\ 1-\gamma+\alpha,\ 1-\beta+\alpha;\frac{1}{z}\right)\\ &+\quad \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F\left(\beta,\ 1-\gamma+\beta,\ 1-\alpha+\beta;\frac{1}{z}\right).\end{aligned}$$

We then calculate

$$\lim_{t\to+\infty}t^{\frac{a}{2}}A(t)=\frac{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{2+a}{4}\right)}+C_2e^{\frac{i\pi(1-a)}{2}}\frac{\Gamma\left(\frac{3-a}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma^2\left(\frac{4-a}{4}\right)}.$$

To make this limit equal to zero we choose

$$C_2=-e^{-\frac{i\pi(1-a)}{2}}\frac{\Gamma\left(\frac{1+a}{2}\right)\Gamma^2\left(\frac{4-a}{4}\right)}{\Gamma^2\left(\frac{2+a}{4}\right)\Gamma\left(\frac{3-a}{2}\right)}.$$

Combining this with (2.7) we conclude

$$\bar{d}_s=\frac{(1-a)\Gamma\left(\frac{1+a}{2}\right)\Gamma^2\left(\frac{4-a}{4}\right)}{\Gamma^2\left(\frac{2+a}{4}\right)\Gamma\left(\frac{3-a}{2}\right)}=\frac{2s\Gamma\left(1-s\right)\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)\Gamma\left(1+s\right)}. \tag{2.8}$$

At this point both constants C_1, C_2, in (2.6) have been identified. After some lengthy but straightforward calculations we find that as $t\to+\infty$

$$A(t)\sim t^{-\frac{2+a}{2}},\qquad A'(t)\sim t^{-\frac{4+a}{2}}\ . \tag{2.9}$$

In addition we get

$$\lim_{t\to+\infty}\frac{tA'(t)}{A(t)}=-\frac{2+a}{2}\ .$$

Using (2.4) and the above asymptotics, we easily conclude that the solution A is energetic, that is,

$$\int_0^\infty t^a(1+t^2)(A')^2dt+\int_0^\infty t^aA^2dt<\infty\ .$$

Multiplying (2.4) by A and integrating by parts in $(0,\infty)$ we arrive at (2.5)

To prove the positivity and monotonicity of A we next change variables by:

$$B(s) = (1+t^2)^{\frac{a}{4}} A(t), \qquad s = 1/t \,.$$

It follows that B satisfies the equation

$$(1+s^2)^2 B'' + (2-a)s(1+s^2)B' - \frac{a^2}{4} B = 0, \qquad s \in (0,+\infty) \,,$$

with $B(0) = 0$ and $B(+\infty) = 1$. A standard maximum principle argument shows that B is positive. Consequently A is positive and the monotonicity of A follows easily.

The positivity and monotonicity of A in connection with the asymptotics of A yield easily part (ii) of the Proposition.

Part (iv) follows easily from the monotonicity of A and part (ii).

□

Using the asymptotics of $A(t)$, from the previous Proposition we easily obtain the following uniform asymptotics for ϕ

Lemma 2.3. *Suppose $a \in (-1,1)$ and let ϕ be given by*

$$\phi(x,y) = d^{-\frac{a}{2}}(x) A\left(\frac{y}{d(x)}\right), \qquad y > 0, \quad x \in \Omega \subset I\!R^n \,,$$

where A solves (2.2), (2.3).
(i) *Then*

$$\phi(x,y) \sim \frac{d}{(d^2+y^2)^{\frac{2+a}{4}}} \,, \quad y > 0, \quad x \in \Omega \,.$$

Concerning the gradient of ϕ, for $a \in (-1,0]$ we have

$$|\nabla_{(x,y)}\phi(x,y)| \sim \frac{1}{(d^2+y^2)^{\frac{2+a}{4}}} \,, \quad y > 0, \quad x \in \Omega \,,$$

whereas for $a \in (0,1)$

$$|\nabla_{(x,y)}\phi(x,y)| \sim \frac{y^{-a}}{(d^2+y^2)^{\frac{2-a}{4}}} \,, \quad y > 0, \quad x \in \Omega \,.$$

(ii) *If Ω satisfies $-\Delta d(x) \geq 0$ for $x \in \Omega$, then for $a \in (-1,0)$*

$$-\mathrm{div}(y^a \nabla\phi)\phi \sim \frac{y^a}{(d^2+y^2)^{\frac{2+a}{2}}}(-d\Delta d) \,, \quad y > 0, \quad x \in \Omega \,,$$

whereas for $a = 0$,

$$-\mathrm{div}(\nabla\phi)\phi \sim \frac{y}{(d^2+y^2)^{\frac{3}{2}}}(-d\Delta d) \,, \quad y > 0, \quad x \in \Omega \,.$$

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 part (i) and (ii): We assume that $s \in [\frac{1}{2}, 1)$ or equivalently $a \in (-1,0]$. We will use Lemma 2.1 with the test function ϕ given by

$$\phi(x,y) = d^{-\frac{a}{2}}(x) A\left(\frac{y}{d(x)}\right), \qquad y > 0, \quad x \in \Omega \subset I\!R^n \,,$$

where A solves (2.2), (2.3). Using Proposition 2.2 and Lemma 2.3 we see that all hypotheses of Lemma 2.1 are satisfied. In particular, for $t=\frac{y}{d}$ we compute, for $x\in\Omega$,

$$\begin{aligned}-\lim_{y\to 0^+}\left(y^a\frac{\phi_y}{\phi}\right) &= -\lim_{y\to 0^+}\left(t^a\frac{A'(t)}{d^{1-a}A(t)}\right)=\frac{1}{d^{1-a}(x)}\lim_{t\to 0^+}\left(-\frac{t^aA'(t)}{A(t)}\right)\\ &= \frac{\bar{d}_s}{d^{1-a}(x)}\,. \end{aligned}\tag{2.10}$$

We also have

$$\begin{aligned}-\mathrm{div}(y^a\nabla\phi) &= -y^{a-1}d^{-1-\frac{a}{2}}\left[(t^3+t)A''+(a+t^2(2+a))A'+\frac{(2+a)a}{4}tA\right]\\ &\quad -y^{a-1}d^{-1-\frac{a}{2}}\left[(-d\Delta d)\left(t^2A'+\frac{at}{2}A\right)\right]\\ &= -y^{a-1}d^{-1-\frac{a}{2}}\left[(-d\Delta d)\left(t^2A'+\frac{at}{2}A\right)\right]\,,\end{aligned}$$

therefore,

$$-\mathrm{div}(y^a\nabla\phi)\geq 0\,,\qquad x\in\Omega,\quad y>0.$$

From Lemma 2.1 we get

$$\begin{aligned}\int_0^{+\infty}\int_\Omega y^a|\nabla u|^2dxdy\geq\qquad &\bar{d}_s\int_\Omega\frac{u^2(x,0)}{d^{1-a}(x)}dx+\int_0^{+\infty}\int_\Omega y^a|\nabla u-\frac{\nabla\phi}{\phi}u|^2dxdy\\ -&\int_0^{+\infty}\int_\Omega\frac{\mathrm{div}(y^a\nabla\phi)}{\phi}u^2dxdy\,,\end{aligned}\tag{2.11}$$

from which the trace Hardy inequality follows directly. This relation will be used later on, in Sections 5 and 6 to obtain the Sobolev term as well.

We continue with the proof of the optimality of the Hardy constant $\bar{d}_s$. Let

$$Q[u]:=\frac{\int_0^{+\infty}\int_\Omega y^a|\nabla u|^2dxdy}{\int_\Omega\frac{u^2(x,0)}{d^{1-a}(x)}dx}=:\frac{N[u]}{D[u]}.\tag{2.12}$$

We have that $Q[u]\geq\bar{d}_s$. Here we will show that there exists a sequence of functions u_ε such that $\lim_{\varepsilon\to 0}Q[u_\varepsilon]=\bar{d}_s$, and therefore $\bar{d}_s$ is the best constant.

We first assume for simplicity that the boundary of Ω is flat in a neighborhood V of a point $x_0\in\partial\Omega$. The neighborhood of the point x_0 is assumed to contain a ball centered at x_0 with radius, say, 3δ. Locally around x_0 the boundary is given by $x_n=0$, whereas the interior of Ω corresponds to $x_n>0$. We also write $x=(x',x_n)$. Clearly, for $x\in\Omega\cap V$ we have that $d(x)=x_n$.

We next define two suitable cutoff functions. Let $\psi(x')\in C_0^\infty(B_\delta)$, where $B_\delta\subset\partial\Omega\subset I\!R^{n-1}$ is the ball centered at x_0 with radius δ. Also the nonnegative function $h(x_n)\in C^\infty(I\!R^+)$ is such that $h(x_n)=0$ for $x_n\geq 2\delta$ and $h(x_n)=1$ for $0\leq x_n\leq\delta$. We will use the following test function:

$$u_\varepsilon(x',x_n,y)=\begin{cases} h(x_n)\psi(x')x_n^{-\frac{a}{2}}A(\frac{y}{x_n}), & y\geq\varepsilon\\ h(x_n)\psi(x')x_n^{-\frac{a}{2}}A(\frac{\varepsilon}{x_n}), & 0\leq y<\varepsilon.\end{cases}\tag{2.13}$$

We have that

$$Q[u_\varepsilon]=\frac{\int_0^{+\infty}dy\int_0^{2\delta}dx_n\int_{B_\delta}dx'y^a|\nabla u_\varepsilon|^2}{\int_0^{2\delta}dx_n\int_{B_\delta}dx'\frac{u_\varepsilon^2}{x_n^{1-a}}}=\frac{N[u_\varepsilon]}{D[u_\varepsilon]}.\tag{2.14}$$

Concerning the denominator we compute

$$\begin{aligned} D[u_\varepsilon] &= \int_{B_\delta} \psi^2(x')dx' \int_0^\delta x_n^{-1} A^2(\frac{\varepsilon}{x_n})dx_n + O_\varepsilon(1) \\ &= \int_{B_\delta} \psi^2(x')dx' \int_{\varepsilon/\delta}^{+\infty} \frac{A^2(t)}{t} dt + O_\varepsilon(1). \end{aligned} \tag{2.15}$$

We next calculate the numerator. At first we break N into two pieces:

$$N[u_\varepsilon] = \int_0^\varepsilon dy + \int_\varepsilon^{+\infty} dy =: N_1[u_\varepsilon] + N_2[u_\varepsilon].$$

Using the specific form of u_ε and elementary estimates we calculate:

$$\begin{aligned} N_2[u_\varepsilon] &= \int_{B_\delta} \psi^2(x')dx' \int_\varepsilon^{+\infty} dy \int_0^\delta dx_n \, \frac{y^a}{x_n^{a+2}} \left[\left(-\frac{a}{2} A(\frac{y}{x_n}) - \frac{y}{x_n} A'(\frac{y}{x_n}) \right)^2 + A'^2(\frac{y}{x_n}) \right] \\ &+ \int_{B_\delta} |\nabla\psi(x')|^2 dx' \int_\varepsilon^{+\infty} dy \int_0^\delta dx_n \, y^a x_n^{-a} A^2(\frac{y}{x_n}) + O_\varepsilon(1) \\ &=: N_{21}[u_\varepsilon] + N_{22}[u_\varepsilon] + O_\varepsilon(1). \end{aligned}$$

We note that as $\varepsilon \to 0$,

$$\begin{aligned} N_{22}[u_\varepsilon] &= \int_{B_\delta} |\nabla\psi(x')|^2 dx' \int_0^\delta x_n \int_{\varepsilon/x_n}^{+\infty} t^a A^2(t) dt dx_n \\ &= O_\varepsilon(1). \end{aligned}$$

Concerning $N_{21}[u_\varepsilon]$, changing variables by $t = \frac{y}{x_n}$ we write:

$$\begin{aligned} N_{21}[u_\varepsilon] &= \int_{B_\delta} \psi^2(x')dx' \int_\varepsilon^{+\infty} \frac{dy}{y} \int_{y/\delta}^{+\infty} \left[t^a A'^2(t) + t^a \left(\frac{a}{2} A(t) + tA'(t) \right)^2 \right] dt \\ &= \int_{B_\delta} \psi^2(x')dx' \int_\varepsilon^{+\infty} \frac{dy}{y} \int_{y/\delta}^{+\infty} \left[t^a(1+t^2)A'^2 + at^{1+a}AA' + \frac{a^2}{4} t^a A^2 \right] dt. \end{aligned}$$

Integrating by parts the term containing the factors AA' and then using the equation satisfied by A (cf (2.4)) we get

$$\begin{aligned} &\int_{y/\delta}^{+\infty} \left[t^a(1+t^2)A'^2 + at^{1+a}AA' + \frac{a^2}{4} t^a A^2 \right] dt \\ &= \int_{y/\delta}^{+\infty} \left[t^a(1+t^2)A'^2 - \frac{a(2+a)}{4} t^a A^2 \right] dt + \frac{1}{2} at^{1+a} A^2(t)|_{t=\frac{y}{\delta}} \\ &= -t^a(1+t^2)A(t)A'(t)|_{t=\frac{y}{\delta}} + \frac{1}{2} at^{1+a} A^2(t)|_{t=\frac{y}{\delta}}, \end{aligned}$$

whence,

$$N_{21}[u_\varepsilon] = -\int_{B_\delta} \psi^2(x')dx' \int_{\varepsilon/\delta}^{+\infty} \frac{1}{t} t^a (1+t^2) A(t)A'(t) dt + O_\varepsilon(1).$$

It is not difficult to show that $N_1[u_\varepsilon] = O_\varepsilon(1)$, and therefore $N[u_\varepsilon] = N_{21}[u_\varepsilon] + O_\varepsilon(1)$. Using also (2.15)

we can form the quotient

$$
\begin{aligned}
\lim_{\varepsilon\to 0} Q[u_\varepsilon] &= \lim_{\varepsilon\to 0} \frac{-\int_{B_\delta}\psi^2(x')dx' \int_{\varepsilon/\delta}^{+\infty}\frac{1}{t}t^a(1+t^2)A(t)A'(t)dt + O_\varepsilon(1)}{\int_{B_\delta}\psi^2(x')dx' \int_{\varepsilon/\delta}^{+\infty}\frac{A^2(t)}{t}dt + O_\varepsilon(1)} \\
&= \lim_{\varepsilon\to 0} \frac{-\int_{\varepsilon/\delta}^{+\infty}\frac{1}{t}t^a(1+t^2)A(t)A'(t)dt}{\int_{\varepsilon/\delta}^{+\infty}\frac{A^2(t)}{t}dt} \\
&= -\lim_{\sigma\to 0}\frac{\sigma^a(1+\sigma^2)A'(\sigma)}{A(\sigma)} \\
&= \bar{d}_s, \qquad (2.16)
\end{aligned}
$$

where we used L'Hopital's rule and then part (i) of Proposition 2.2.

Let us now consider the general case. We assume that $\partial\Omega$ is C^1 in a neighborhood of a point $\bar{x}_0$, which we take to be the origin $0\in\partial\Omega$. Thus locally $\partial\Omega$, is the graph of a function $\bar{x}_n=\gamma(\bar{x}')$, with $\gamma(0)=0$ and $\nabla\gamma(0)=0$. We also assume that the interior of Ω corresponds to $\bar{x}_n>\gamma(\bar{x}')$. Then the following change of coordinates straightens the boundary in a neighborhood of the origin: $x_i=\bar{x}_i$, $i=1,2,\dots,n-1$, and $x_n=\bar{x}_n-\gamma(\bar{x}')$; see e.g. [E], Appendix C. We assume that inside the ball $B(0,3\delta)$ (in the x-space) the image of $\partial\Omega$ is flat. We then consider the test function $v_\varepsilon(\bar{x},y)=u_\varepsilon(x,y)$. Clearly $v_\varepsilon(\bar{x},y)$ is zero away from a neighborhood of the origin, say U, and elementary calculations show that

$$\nabla_{\bar{x}}v_\varepsilon=\nabla_x u_\varepsilon-u_{\varepsilon,x_n}\nabla_{\bar{x}}\gamma(\bar{x}'),$$

whence,

$$|\nabla_{\bar{x}}v_\varepsilon-\nabla_x u_\varepsilon|\le|\nabla_{\bar{x}}\gamma(\bar{x}')||\nabla_x u_\varepsilon|=o_\delta(1)|\nabla_x u_\varepsilon|.$$

It then follows that

$$|\nabla_{\bar{x}}v_\varepsilon|=|\nabla_x u_\varepsilon|(1+o_\delta(1)).$$

On the other hand, for $\bar{x}\in U$ and $d(\bar{x})=\mathrm{dist}(\bar{x},\partial\Omega)$, we have that

$$d(\bar{x})=(\bar{x}_n-\gamma(\bar{x}'))(1+|\nabla_{\bar{x}}\gamma(\bar{x}')|^2)^{1/2}=x_n(1+o_\delta(1)).$$

We finally note that the Jacobian of the above transformation is one and therefore $dx=d\bar{x}$. We then compute

$$Q[v_\varepsilon(\bar{x},y)]=Q[u_\varepsilon(x,y)](1+o_\delta(1)),$$

where $Q[u_\varepsilon(x,y)]$ is given in (2.14). Since δ can be taken as small as we like the result follows easily, using the calculations from the flat case.

□

3 The Trace Hardy inequality II

In this section we will prove the trace Hardy inequality contained in Theorem 1.4. We first establish the analogue of Lemma 2.1:

Lemma 3.1. *Suppose that $a\in(-1,1)$ and let $u\in C_0^\infty(I\!R^n\times I\!R)$ such that $u(\cdot,0)\in C_0^\infty(\Omega)$. Let $\phi\in C^2(I\!R^n\times(0,\infty))\cap C(I\!R^n\times[0,\infty))$ is such that $\phi(x,y)>0$ in $I\!R^n\times[0,\infty)$, $\phi(x,0)=0$ in $x\in\mathcal{C}\Omega$,*

$$|y^a\frac{\phi_y(x,y)}{\phi(x,y)}|\le V(x),\quad y\in(0,1),\quad x\in I\!R^n,\qquad 0\le V(x)\in L^1_{loc}(I\!R^n)\ .$$

Moreover for a.e. $x\in\Omega$, the following limit exists:

$$\lim_{y\to 0^+}\left(y^a\frac{\phi_y(x,y)}{\phi(x,y)}\right)\ .$$

We also require that the following integrals are finite

$$\int_0^{+\infty}\int_{I\!R^n} y^a\frac{|\nabla\phi|^2}{\phi^2}u^2dxdy, \qquad \int_0^{+\infty}\int_{I\!R^n}\frac{|\mathrm{div}(y^a\nabla\phi)|}{\phi}u^2dxdy\ .$$

We then have the identity:

$$\begin{aligned}\int_0^{+\infty}\int_{I\!R^n} y^a|\nabla u|^2dxdy = &-\int_\Omega\lim_{y\to 0+}\left(y^a\frac{\phi_y}{\phi}\right)u^2(x,0)dx+\int_0^{+\infty}\int_{I\!R^n}y^a|\nabla u-\frac{\nabla\phi}{\phi}u|^2dxdy\\ &-\int_0^{+\infty}\int_{I\!R^n}\frac{\mathrm{div}(y^a\nabla\phi)}{\phi}u^2dxdy. \end{aligned}\tag{3.1}$$

The proof of this Lemma is quite similar to the proof of Lemma 2.1 and we omit it.

This time we will choose the test function to be of the form

$$\phi(x,y)=\begin{cases}(y^2+d^2)^{-\frac{a}{4}}B(\frac{d}{y}), & x\in\Omega,\quad y>0\\ (y^2+d^2)^{-\frac{a}{4}}B(-\frac{d}{y}), & x\in\mathcal{C}\Omega,\ \ y>0\end{cases}\tag{3.2}$$

where function B is the solution of the following boundary value problem

$$(1+t^2)^2B''+(2-a)t(1+t^2)B'-\frac{a^2}{4}B=0, \qquad t\in(-\infty,+\infty)\ ,\tag{3.3}$$

complemented with the conditions

$$B(-\infty)=0, \qquad B(+\infty)=1.\tag{3.4}$$

We note that this can be written in divergence form as

$$((1+t^2)^{1-\frac{a}{2}}B'(t))'-\frac{a^2}{4}(1+t^2)^{-1-\frac{a}{2}}B(t)=0, \quad t\in I\!R.\tag{3.5}$$

We next collect some properties of B that will be used later on.

Proposition 3.2. *Suppose that $a\in(-1,1)$. The boundary value problem (3.3), (3.4) has a positive increasing solution B with the following properties:*
(i) There exists a positive constant $\bar{k}_s$ such that

$$\lim_{t\to+\infty}(1+t^2)^{\frac{2-a}{2}}B'(t)=:\bar{k}_s\ ,\tag{3.6}$$

where

$$\bar{k}_s=\frac{2^a\Gamma^2(\frac{2-a}{2})\Gamma(\frac{1+a}{2})}{\pi\Gamma(\frac{1-a}{2})}=\frac{2^{1-2s}}{\pi}\frac{\Gamma^2(s+\frac12)\Gamma(1-s)}{\Gamma(s)}\ .$$

(ii) We have

$$\begin{aligned} B(t) &\sim 1, & t>0\\ B(t) &\sim (1+t^2)^{-\frac{1-a}{2}} & t<0\ ,\\ B'(t) &\sim (1+t^2)^{-\frac{2-a}{2}} & t\in I\!R\ .\end{aligned}$$

(iii) There holds:

$$\bar{k}_s=\int_{-\infty}^{+\infty}\left[(1+t^2)^{1-\frac{a}{2}}B'^2(t)+\frac{a^2}{4}(1+t^2)^{-1-\frac{a}{2}}B^2(t)\right]dt\ .$$

(iv) In case $a\in(-1,0]$, we have

$$(1+t^2)B'(t)-\frac{a}{2}tB(t)>0, \qquad t\in I\!R\ .$$

Moreover for $a\in(-1,0)$

$$(1+t^2)B'(t)-\frac{a}{2}tB(t)\sim(1+t^2)^{\frac12}, \qquad t>0\ .$$

Proof: When $a = 0$ the ODE can be easily solved by a straightforward integration. For the general case we first change variables by $B(t) = (1+t^2)^{\frac{a}{4}} f(t)$ to obtain

$$(1+t^2)f'' + 2tf' + \frac{a(2-a)}{4}f = 0.$$

We next change variables by $g(z) = f(t)$, $z = it$, so that g satisfies the equation

$$(1-z^2)g'' - 2zg' + \nu(\nu+1)g = 0, \qquad \nu = -\frac{a}{2}. \tag{3.7}$$

The solution of this is given in [AS], Section 8.1:

$$g(z) = \begin{cases} C_1^+ P_\nu(z) + C_2^+ Q_\nu(z), & \mathrm{Im} z > 0, \\ C_1^- P_\nu(z) + C_2^- Q_\nu(z), & \mathrm{Im} z < 0. \end{cases} \tag{3.8}$$

We also have that

$$B(t) = (1+t^2)^{-\frac{\nu}{2}} g(it).$$

The conditions then at infinity become

$$\lim_{t\to+\infty} t^{-\nu} g(it) = 1, \qquad \lim_{t\to-\infty} (-t)^{-\nu} g(it) = 0. \tag{3.9}$$

To find the constants in (3.8) we will satisfy the conditions at infinity (3.9) and we will match both g and g' at $z = 0$. That is we will ask

$$g(+i0) = g(-i0), \qquad g'(+i0) = g'(-i0). \tag{3.10}$$

We recall from [AS] Section 8.1 that for $|z| > 1$:

$$\begin{aligned} P_\nu(z) &= \Delta_1 z^{-\nu-1} F\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}, \frac{2\nu+3}{2}; \frac{1}{z^2}\right) + \Delta_2 z^\nu F\left(\frac{-\nu}{2}, \frac{1-\nu}{2}, \frac{1-2\nu}{2}; \frac{1}{z^2}\right), \\ Q_\nu(z) &= E_1 z^{-\nu-1} F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}, \frac{2\nu+3}{2}; \frac{1}{z^2}\right). \end{aligned}$$

where,

$$\Delta_1 = \frac{2^{-\nu-1}\pi^{-\frac{1}{2}}\Gamma(-\nu-\frac{1}{2})}{\Gamma(-\nu)}, \quad \Delta_2 = \frac{2^\nu \pi^{-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}{\Gamma(1+\nu)}, \quad E_1 = \frac{2^{-\nu-1}\pi^{\frac{1}{2}}\Gamma(1+\nu)}{\Gamma(\frac{3}{2}+\nu)}.$$

From the asymptotics when $t \to \pm\infty$, we easily conclude that

$$C_1^+ = \frac{i^{-\nu}}{\Delta_2}, \qquad C_1^- = 0. \tag{3.11}$$

We next see what happens near zero. For $|z| < 1$ we have that

$$\begin{aligned} P_\nu(z) &= B_1 F\left(-\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1}{2}; z^2\right) + B_2 z F\left(\frac{1-\nu}{2}, \frac{2+\nu}{2}, \frac{3}{2}; z^2\right), \\ Q_\nu^\pm(z) &= \Gamma_1 e^{\pm\frac{i\pi}{2}(-\nu-1)} F\left(-\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1}{2}; z^2\right) + \Gamma_2 e^{\pm\frac{i\pi}{2}(-\nu)} z F\left(\frac{1-\nu}{2}, \frac{\nu+2}{2}, \frac{3}{2}; z^2\right), \end{aligned}$$

where the plus sign corresponds to $\mathrm{Im} z > 0$ and the minus to $\mathrm{Im} z < 0$. The value of the constants are given by:

$$B_1 = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\nu}{2})\Gamma(\frac{2+\nu}{2})}, \;\; B_2 = \frac{-2\pi^{\frac{1}{2}}}{\Gamma(\frac{1+\nu}{2})\Gamma(\frac{-\nu}{2})}, \;\; \Gamma_1 = \frac{\pi^{\frac{1}{2}}\Gamma(\frac{1+\nu}{2})}{2\Gamma(1+\frac{\nu}{2})}, \;\; \Gamma_2 = \frac{\pi^{\frac{1}{2}}\Gamma(1+\frac{\nu}{2})}{\Gamma(\frac{1+\nu}{2})}.$$

An easy calculation shows that the matching condition (3.10) yields

$$\begin{aligned} C_2^- \Gamma_1 e^{\frac{i\pi}{2}(\nu+1)} &= C_1^+ B_1 + C_2^+ \Gamma_1 e^{\frac{i\pi}{2}(-\nu-1)}, \\ C_2^- \Gamma_2 e^{\frac{i\pi}{2}\nu} &= C_1^+ B_2 + C_2^+ \Gamma_2 e^{\frac{i\pi}{2}(-\nu)}, \end{aligned}$$

from which it follows that

$$\begin{aligned} C_2^+ &= -\frac{C_1^+}{2} e^{\frac{i\pi}{2}\nu} \left[\frac{B_2}{\Gamma_2} + i \frac{B_1}{\Gamma_1} \right] \\ C_2^- &= \frac{C_1^+}{2} e^{-\frac{i\pi}{2}\nu} \left[\frac{B_2}{\Gamma_2} - i \frac{B_1}{\Gamma_1} \right]. \end{aligned} \tag{3.12}$$

Thus all constants in (3.8) have been computed (cf (3.11) and (3.12)), and therefore $g(z)$ is now completely known.

The asymptotics of g for $|z| \to +\infty$, are

$$\begin{aligned} g(z) &= C_1^\pm \Delta_2 z^\nu + (C_1^\pm \Delta_1 + C_2^\pm E_1) z^{-\nu-1} + o(|z|^{-\nu-1}), \\ g'(z) &= C_1^\pm \Delta_2 \nu z^{\nu-1} - (\nu+1)[C_1^\pm \Delta_1 + C_2^\pm E_1] z^{-\nu-2} + O(|z|^{\nu-3}), \end{aligned}$$

where the plus sign corresponds to $\mathrm{Im} z > 0$ and the minus to $\mathrm{Im} z < 0$. We have that $B(t) = (1+t^2)^{-\frac{\nu}{2}} g(it)$, whence we get

$$B(t) = i^{1+\nu} C_2^- E_1 (-t)^{-2\nu-1} + o((-t)^{-2\nu-1}), \qquad t \to -\infty.$$

Concerning the derivative, we have for $z = it$

$$B'(t) = -\nu t (t^2+1)^{-\frac{\nu}{2}-1} g(z) + i(1+t^2)^{-\frac{\nu}{2}} g'(z).$$

Whence,

$$\begin{aligned} B'(t) &= (2\nu+1) i^{1-\nu} (C_1^+ \Delta_1 + C_2^+ E_1)\, t^{-2\nu-2} + o(t^{-2\nu-2}), \quad t \to +\infty, \\ B'(t) &= (2\nu+1) i^{1+\nu} C_2^- E_1\, (-t)^{-2\nu-2} + o((-t)^{-2\nu-2}), \quad t \to -\infty\,. \end{aligned}$$

This completes the proof of part (ii) of the Proposition.

We next give the proof of part (i). From (3.6) and the asymptotics of $B(t)$ for $t \to +\infty$, we compute

$$\bar{k}_s = \frac{(2\nu+1)}{2} i^{1-2\nu} \frac{E_1}{\Delta_2} \left(2\frac{\Delta_1}{E_1} - i^\nu \frac{B_2}{\Gamma_2} - i^{\nu+1} \frac{B_1}{\Gamma_1} \right). \tag{3.13}$$

Using the explicit values of the constants we calculate:

$$\frac{E_1}{\Delta_2} = \frac{2^{-2\nu-1}\, \pi\, \Gamma^2(1+\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{3}{2}+\nu)}, \quad \frac{\Delta_1}{E_1} = \frac{\sin(\pi\nu)}{\pi\cos(\pi\nu)}, \quad \frac{B_2}{\Gamma_2} = \frac{2\sin(\frac{\pi\nu}{2})}{\pi}, \quad \frac{B_1}{\Gamma_1} = \frac{2\cos(\frac{\pi\nu}{2})}{\pi}.$$

Plugging these in (3.13) we conclude that (recall that $\nu = -a/2 = s - 1/2$)

$$\bar{k}_s = \frac{2^{-2\nu}}{\pi} \frac{\Gamma^2(1+\nu)\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)} = \frac{2^a \Gamma^2(\frac{2-a}{2})\Gamma(\frac{1+a}{2})}{\pi\Gamma(\frac{1-a}{2})} = \frac{2^{1-2s}}{\pi} \frac{\Gamma^2(s+\frac{1}{2})\Gamma(1-s)}{\Gamma(s)}. \tag{3.14}$$

To prove part (iii) we use part (i) and we integrate the ODE (3.5).

By standard maximum principle arguments the solution $B(t)$ of (3.3) subject to (3.4) is positive and increasing. To prove part (iv) assuming that $a \in (-1, 0)$, we set $f(t) = (1+t^2)^{-\frac{a}{4}} B(t)$ so that

$$(1+t^2) f'' + 2t f' + \frac{a(2-a)}{4} f = 0,$$

and a similar maximum principle argument shows that $f(t)$ is also increasing. Since,

$$f'(t) = (1+t^2)^{-\frac{a}{4}-1}\left[(1+t^2)B' - \frac{a}{2}tB\right],$$

we conclude that

$$(1+t^2)B' - \frac{a}{2}tB > 0, \qquad t \in I\!R, \qquad a \le 0.$$

Using the asymptotics of B, B' from part (ii) we conclude the proof of part (iv).

$\square$

Using the asymptotics of $B(t)$ from the previous Proposition, we easily obtain the following uniform asymptotics for ϕ

Lemma 3.3. *Suppose $a \in (-1,1)$ and let ϕ be given by*

$$\phi(x,y) = \begin{cases} (y^2+d^2)^{-\frac{a}{4}}B(\frac{d}{y}), & x \in \Omega, \quad y > 0 \\ (y^2+d^2)^{-\frac{a}{4}}B(-\frac{d}{y}), & x \in \mathcal{C}\Omega, \ \ y > 0 \,, \end{cases}$$

where B solves (3.3), (3.4).
(i) *Then*

$$\phi(x,y) \sim \begin{cases} (y^2+d^2)^{-\frac{a}{4}}, & x \in \Omega, \quad y > 0 \\ y^{1-a}(y^2+d^2)^{\frac{a-2}{4}}, & x \in \mathcal{C}\Omega, \ \ y > 0. \end{cases}$$

Concerning the gradient of ϕ, for $a \in (-1,0]$ we have

$$|\nabla\phi(x,y)| \sim \begin{cases} (y^2+d^2)^{-\frac{a+2}{4}}, & x \in \Omega, \quad y > 0 \\ y^{-a}(y^2+d^2)^{\frac{a-2}{4}}, & x \in \mathcal{C}\Omega, \ \ y > 0. \end{cases}$$

whereas for $a \in (0,1)$

$$|\nabla\phi(x,y)| \sim y^{-a}(y^2+d^2)^{\frac{a-2}{4}}, \qquad x \in I\!R^n, \quad y > 0\,.$$

(ii) *If Ω satisfies $-\Delta d(x) \ge 0$ for $x \in \Omega$, then for $a \in (-1,0)$*

$$-\mathrm{div}(y^a\nabla\phi)\phi \sim \frac{y^a}{d(d^2+y^2)^{\frac{1+a}{2}}}(-d\Delta d)\,, \quad y > 0, \quad x \in \Omega\,,$$

whereas for $a = 0$,

$$-\mathrm{div}(\nabla\phi)\phi \sim \frac{y}{d(d^2+y^2)}(-d\Delta d)\,, \quad y > 0, \quad x \in \Omega\,.$$

We are now ready to give the proof of Theorem 1.4
Proof of Theorem 1.4 part (i) and (ii): We assume that $s \in [\frac{1}{2},1)$ or equivalently $a \in (-1,0]$. We will use Lemma 3.1 with the test function ϕ given

$$\phi(x,y) = \begin{cases} (y^2+d^2)^{-\frac{a}{4}}B(\frac{d}{y}), & x \in \Omega, \quad y > 0 \\ (y^2+d^2)^{-\frac{a}{4}}B(-\frac{d}{y}), & x \in \mathcal{C}\Omega, \ \ y > 0 \,, \end{cases}$$

Using Proposition 3.2 and Lemma 3.3 we see that all hypotheses of Lemma 3.1 are satisfied. In particular we compute

$$\begin{aligned} -\lim_{y\to 0^+}\left(y^a\frac{\phi_y(x,y)}{\phi(x,y)}\right) &= \frac{1}{d^{1-a}(x)}\lim_{t\to+\infty}\left(t^{2-a}B'(t)\right) \\ &= \frac{\bar{k}_s}{d^{1-a}(x)}\,, \qquad x \in \Omega\,. \end{aligned} \tag{3.15}$$

We also have for $x \in \Omega$ and $t = \frac{d}{y} > 0$,

$$\begin{aligned} -\operatorname{div}(y^a \nabla \phi) &= -y^a (y^2+d^2)^{-\frac{a}{4}-1} \left[(1+t^2)^2 B'' + (2-a)t(1+t^2)B' - \frac{a^2}{4} B \right] \\ &\quad + y^{a+1}(y^2+d^2)^{-\frac{a}{4}-1}(-\Delta d) \left[(1+t^2)B' - \frac{a}{2} tB \right] \\ &= y^{a+1}(y^2+d^2)^{-\frac{a}{4}-1}(-\Delta d) \left[(1+t^2)B' - \frac{a}{2} tB \right] , \end{aligned} \tag{3.16}$$

whereas for $x \in \mathcal{C}\Omega$ and $t = -\frac{d}{y} < 0$, we have

$$\begin{aligned} -\operatorname{div}(y^a \nabla \phi) &= -y^a (y^2+d^2)^{-\frac{a}{4}-1} \left[(1+t^2)^2 B'' + (2-a)t(1+t^2)B' - \frac{a^2}{4} B \right] \\ &\quad + y^{a+1}(y^2+d^2)^{-\frac{a}{4}-1}(\Delta d) \left[(1+t^2)B' - \frac{a}{2} tB \right] \\ &= y^{a+1}(y^2+d^2)^{-\frac{a}{4}-1}(\Delta d) \left[(1+t^2)B' - \frac{a}{2} tB \right] . \end{aligned} \tag{3.17}$$

Therefore under our assumption on Ω it follows from Proposition 3.2 that

$$-\operatorname{div}(y^a \nabla \phi) \geq 0, \qquad x \in I\!\!R^n, \quad y > 0 .$$

We now use Lemma 3.1 to get

$$\begin{aligned} \int_0^{+\infty} \int_{I\!\!R^n} y^a |\nabla u|^2 dx dy \geq \qquad & \bar{k}_s \int_\Omega \frac{u^2(x,0)}{d^{1-a}(x)} dx + \int_0^{+\infty} \int_{I\!\!R^n} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dx dy \\ - & \int_0^{+\infty} \int_{I\!\!R^n} \frac{\operatorname{div}(y^a \nabla \phi)}{\phi} u^2 dx dy , \end{aligned} \tag{3.18}$$

from which the trace Hardy inequality follows directly. This relation will also be used later on, in Section 5 and 6 to obtain the Sobolev term as well.

We next prove the optimality of the Hardy constant. We will work as in section 2. Let

$$Q[u] := \frac{\int_0^{+\infty} \int_{I\!\!R^n} y^a |\nabla u|^2 dx dy}{\int_\Omega \frac{u^2(x,0)}{d^{1-a}(x)} dx} =: \frac{N[u]}{D[u]}. \tag{3.19}$$

We will show that there exists a sequence of functions u_ε such that $\lim_{\varepsilon \to 0} Q[u_\varepsilon] \leq \bar{k}_s$, and therefore $\bar{k}_s$ is the best constant.

We first assume that the boundary of Ω is flat in a neighborhood U of a point $x_0 \in \partial\Omega$. The neighborhood of the point x_0 is assumed to contain a ball centered at x_0 with radius, say, 3δ. Locally around x_0 the boundary is given by $x_n = 0$, whereas the interior of Ω corresponds to $x_n > 0$. We also write $x = (x', x_n)$. Clearly, for $x \in \Omega \cap U$ we have that $d(x) = x_n$.

We next define three suitable cutoff functions. Let $\psi(x') \in C_0^\infty(B_\delta)$, where $B_\delta \subset \partial\Omega \subset I\!\!R^{n-1}$ is the ball centered at x_0 with radius δ. Also the nonnegative function $h(x_n) \in C^\infty(I\!\!R)$ is such that $h(x_n) = 0$ for $|x_n| \geq 2\delta$ and $h(x_n) = 1$ for $|x_n| \leq \delta$. We also assume that $h(x_n)$ is symmetric around $x_n = 0$. Finally let $\chi(y) \in C_0^\infty(I\!\!R)$ be such that $0 \leq \chi(y) \leq 1$, and $\chi(y) = 1$ near $y = 0$.

We will use the following test function:

$$u_\varepsilon(x', x_n, y) = \chi(y) h(x_n) \psi(x') (y^2 + x_n^2)^{-\frac{a}{4} + \frac{\varepsilon}{4}} B(\frac{x_n}{y}), \qquad x \in I\!\!R^n, \quad y > 0. \tag{3.20}$$

Using the asymptotics of $B(t)$ we easily see that

$$u_\varepsilon(x', x_n, 0) = \begin{cases} h(x_n)\psi(x') x_n^{-\frac{a}{2} + \frac{\varepsilon}{2}}, & x \in \Omega \\ 0, & x \in \mathcal{C}\Omega. \end{cases}$$

We then compute

$$D[u_\varepsilon] = \int_{I\!R^{n-1}} \psi^2(x')dx' \int_0^{+\infty} h^2(x_n)x_n^{-1+\varepsilon}dx_n. \tag{3.21}$$

Concerning the numerator, a straightforward calculation shows that

$$\begin{aligned}|\nabla((y^2+x_n^2)^{-\frac{a}{4}+\frac{\varepsilon}{4}}B(\frac{x_n}{y}))|^2 = & \left(-\frac{a}{2}+\frac{\varepsilon}{2}\right)^2 (y^2+x_n^2)^{-\frac{a}{2}+\frac{\varepsilon}{2}-1}B^2(\frac{x_n}{y}) \\ & + \frac{(x_n^2+y^2)^{1-\frac{a}{2}+\frac{\varepsilon}{2}}}{y^4}B'^2(\frac{x_n}{y}).\end{aligned}$$

It is then easy to show that

$$\begin{aligned}N[u_\varepsilon] = & \int_{I\!R^{n-1}} \psi^2(x')dx' \int_{I\!R}\int_0^{+\infty} h^2(x_n)y^a\chi^2(y)\left[\left(-\frac{a}{2}+\frac{\varepsilon}{2}\right)^2 (y^2+x_n^2)^{-\frac{a}{2}+\frac{\varepsilon}{2}-1}B^2(\frac{x_n}{y})\right. \\ & \left. + \frac{(x_n+y^2)^{1-\frac{a}{2}+\frac{\varepsilon}{2}}}{y^4}B'^2(\frac{x_n}{y})\right] dydx_n + O_\varepsilon(1).\end{aligned}$$

To estimate the double integral above, we first break the x_n–integral into two pieces: from minus infinity to zero and from zero to infinity. We then change variables in both pieces by $t = x_n/y$, thus going from the (x_n, y) variables to (x_n, t). After elementary calculations we arrive at

$$\begin{aligned}N[u_\varepsilon] = & \int_{I\!R^{n-1}} \psi^2(x')dx' \int_0^{+\infty} h^2(x_n)x_n^{-1+\varepsilon}dx_n \cdot \\ & \int_{-\infty}^{+\infty} \chi^2\left(\frac{x_n}{|t|}\right)\left[\frac{(1+t^2)^{1-\frac{a}{2}+\frac{\varepsilon}{2}}}{|t|^\varepsilon}B'^2(t) + \left(-\frac{a}{2}+\frac{\varepsilon}{2}\right)^2 \frac{(1+t^2)^{-1-\frac{a}{2}+\frac{\varepsilon}{2}}}{|t|^\varepsilon}B^2(t)\right] dt + O_\varepsilon(1).\end{aligned}$$

Forming the quotient we obtain

$$Q[u_\varepsilon] \le \int_{-\infty}^{+\infty}\left[\frac{(1+t^2)^{1-\frac{a}{2}+\frac{\varepsilon}{2}}}{|t|^\varepsilon}B'^2(t) + \left(-\frac{a}{2}+\frac{\varepsilon}{2}\right)^2 \frac{(1+t^2)^{-1-\frac{a}{2}+\frac{\varepsilon}{2}}}{|t|^\varepsilon}B^2(t)\right] dt + o_\varepsilon(1)$$

We finally send ε to zero to get

$$\begin{aligned}\lim_{\varepsilon\to 0} Q[u_\varepsilon] & \le \int_{-\infty}^{+\infty}\left[(1+t^2)^{1-\frac{a}{2}}B'^2(t) + \frac{a^2}{4}(1+t^2)^{-1-\frac{a}{2}}B^2(t)\right] dt \\ & = \bar{k}_s;\end{aligned} \tag{3.22}$$

the last equality follows from Proposition 3.2(iii).

The general case where $\partial\Omega$ is not flat is treated in the same way as in section 2.

□

4 Some Weighted Hardy Inequalities

In this section we establish some new weighted Hardy inequalities that will play a crucial role in establishing trace Hardy–Sobolev–Maz'ya inequalities.

We first prove the following:

Lemma 4.1. *Let $\Omega \subset I\!R^n$ be such that $-\Delta d(x) \ge 0$ for $x \in \Omega$. If A, B, Γ are constants such that $A+1>0$, $B+1>0$ and $2\Gamma < A+B+2$ then for all $v \in C_0^\infty(I\!R^n \times I\!R)$ there holds*

$$\begin{aligned}\frac{(B+1)(B+A+2-2\Gamma^+)}{B+A+2}\int_0^{+\infty}\int_\Omega \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy \quad & \le \\ \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}(-\Delta d)|v|dxdy \quad + \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|\nabla v|dxdy\ , &\end{aligned} \tag{4.1}$$

where $\Gamma^+ = \max(0,\Gamma)$.

Proof: Integrating by parts in the x-variables we compute

$$
\begin{aligned}
(B+1)\int_0^{+\infty}\int_\Omega \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy &= \int_0^{+\infty}\int_\Omega \frac{y^A \nabla d\cdot\nabla d^{B+1}}{(d^2+y^2)^\Gamma}|v|dxdy \\
= \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}(-\Delta d)}{(d^2+y^2)^\Gamma}|v|dxdy &+ 2\Gamma\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2}}{(d^2+y^2)^{\Gamma+1}}|v|dxdy \\
&- \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}\nabla d\cdot\nabla_x|v|dxdy. \qquad (4.2)
\end{aligned}
$$

If $\Gamma \le 0$ the result follows easily. In the sequel we consider the case $\Gamma > 0$. In the previous calculation there is no boundary term due to our assumptions. To continue we will estimate the middle term in the right hand side above. To this end we define the vector field $\vec{F}$ by

$$
\vec{F}(x,y) := \left(\frac{y^A d^{B+3}\nabla d}{(d^2+y^2)^{\Gamma+1}}, \frac{y^{A+1}d^{B+2}}{(d^2+y^2)^{\Gamma+1}}\right). \qquad (4.3)
$$

We then have

$$
\int_0^{+\infty}\int_\Omega \mathrm{div}\vec{F}|v|dxdy = -\int_0^{+\infty}\int_\Omega \vec{F}\cdot\nabla|v|dxdy \le \int_0^{+\infty}\int_\Omega |\vec{F}||\nabla v|dxdy. \qquad (4.4)
$$

We note that because of our assumptions $A+1>0$ and $B+1>0$, there are no boundary terms in (4.4). Straightforward calculations show that

$$
\mathrm{div}\vec{F} = \frac{y^A d^{B+3}(\Delta d)}{(d^2+y^2)^{\Gamma+1}} + (B+A+2-2\Gamma)\frac{y^A d^{B+2}}{(d^2+y^2)^{\Gamma+1}}, \qquad (4.5)
$$

and

$$
|\vec{F}| = \frac{y^A d^{B+2}}{(d^2+y^2)^{\Gamma+1/2}} \le \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}. \qquad (4.6)
$$

From (4.4)–(4.6) we get

$$
\begin{aligned}
&(B+A+2-2\Gamma)\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2}}{(d^2+y^2)^{\Gamma+1}}|v|dxdy \\
&\le \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+3}}{(d^2+y^2)^{\Gamma+1}}(-\Delta d)|v|dxdy \quad + \quad \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|\nabla v|dxdy.
\end{aligned}
$$

Combining the above with (4.2) we conclude the proof.

□

We will also need a version of the above Lemma in case where $A+B+2 = 2\Gamma$. In this case we have:

Lemma 4.2. *Suppose that $\Omega \subset I\!R^n$ has finite inner radius and is such that $-\Delta d(x) \ge 0$ for $x \in \Omega$. If A, B are constants such that $A+1>0$, $B+1>0$, then for all $v \in C_0^\infty(I\!R^n \times I\!R)$ there holds*

$$
\begin{aligned}
&\frac{B+1}{A+B+3}\int_0^{+\infty}\int_\Omega \frac{y^A d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}|v|dxdy \quad \le \qquad (4.7) \\
&\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}X}{(d^2+y^2)^{\frac{A+B+2}{2}}}(-\Delta d)|v|dxdy \quad + \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}X}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla v|dxdy\,,
\end{aligned}
$$

where $X = X(\frac{d(x)}{R_{in}})$ and $X(t) = (1-\ln t)^{-1}$, $0 < t \le 1$.

Proof: Integrating by parts in the x-variables we compute

$$\begin{aligned}
&(B+1)\int_0^{+\infty}\int_\Omega \frac{y^A d^B X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}|v|dxdy + 2\int_0^{+\infty}\int_\Omega \frac{y^A d^B X^3}{(d^2+y^2)^{\frac{A+B+2}{2}}}|v|dxdy \\
\leq &\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1} X^2(-\Delta d)}{(d^2+y^2)^{\frac{A+B+2}{2}}}|v|dxdy + (A+B+2)\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2} X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}|v|dxdy \\
&+\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1} X^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla v|dxdy. \qquad (4.8)
\end{aligned}$$

In the previous calculation there are no boundary terms due to our assumptions. To continue we will estimate the middle term in the right hand side above. To this end we define the vector field $\vec{F}$ by

$$\vec{F}(x,y) := \left(\frac{y^A d^{B+3} X \nabla d}{(d^2+y^2)^{\frac{A+B+4}{2}}},\ \frac{y^{A+1} d^{B+2} X}{(d^2+y^2)^{\frac{A+B+4}{2}}} \right). \qquad (4.9)$$

We then have

$$\int_0^{+\infty}\int_\Omega \mathrm{div}\vec{F}|v|dxdy = -\int_0^{+\infty}\int_\Omega \vec{F}\cdot\nabla|v|dxdy \leq \int_0^{+\infty}\int_\Omega |\vec{F}||\nabla v|dxdy. \qquad (4.10)$$

We note that because of our assumptions $A+1>0$ and $B+1>0$, there are no boundary terms in (4.10). Straightforward calculations show that

$$\mathrm{div}\vec{F} = \frac{y^A d^{B+3} X(\Delta d)}{(d^2+y^2)^{\frac{A+B+4}{2}}} + \frac{y^A d^{B+2} X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}, \qquad (4.11)$$

and

$$|\vec{F}| = \frac{y^A d^{B+2} X}{(d^2+y^2)^{\frac{A+B+3}{2}}} \leq \frac{y^A d^{B+1} X}{(d^2+y^2)^{\frac{A+B+2}{2}}}. \qquad (4.12)$$

From (4.10)–(4.12) we get

$$\begin{aligned}
&\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+2} X^2}{(d^2+y^2)^{\frac{A+B+4}{2}}}|v|dxdy \\
\leq &\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+3} X}{(d^2+y^2)^{\frac{A+B+4}{2}}}(-\Delta d)|v|dxdy \quad + \quad \int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1} X}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla v|dxdy.
\end{aligned}$$

Combining the above with (4.8) we conclude the proof.

□

Without imposing any geometric assumption on Ω we have the following result that will also be used later on.

Lemma 4.3. *Let $\Omega \subset I\!R^n$. If A, B, Γ are constants such that $A+1>0$, $B+1>0$ and $2\Gamma < A+B+2$, then there exist positive constants c_1 and c_2 such that for all $v \in C_0^\infty(I\!R^n \times I\!R)$ there holds*

$$\begin{aligned}
&\int_0^{+\infty}\int_\Omega \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy \qquad (4.13)\\
&\quad \leq c_1\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|\nabla v|dxdy + c_2\int_0^{+\infty}\int_\Omega \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|v|dxdy\ .
\end{aligned}$$

Proof: Here we will use the fact that $\partial\Omega$ is uniformly Lipschitz. Let $\{U_i\}$ be a covering of $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$ and let ϕ_i be a partition of unity subordinate to the covering $\{U_i\}$. We then have

$$\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy \le \sum_{i=1}^{+\infty}\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|\phi_i v|dd.$$

In each U_i we straighten the boundary and use the equivalence of the distance function to the regularized distance as well as to the difference $x_n - f_i(x')$ (see [St] section 3.2, or [L] section 12.2) and obtain

$$\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|\phi_i v|dxdy \le C\int_0^{+\infty}\int_{I\!R^n_+} \frac{y^A t^B}{(t^2+y^2)^\Gamma}|\tilde{\phi}_i\tilde{v}|dxdy\ ,$$

for some constant C independent of i. We next use Lemma 4.1 to estimate the right hand side of this, thus obtaining

$$\int_0^{+\infty}\int_{I\!R^n_+} \frac{y^A t^B}{(t^2+y^2)^\Gamma}|\tilde{\phi}_i\tilde{v}|dxdy \le C\int_0^{+\infty}\int_{I\!R^n_+} \frac{y^A t^{B+1}}{(t^2+y^2)^\Gamma}|\nabla(\tilde{\phi}_i\tilde{v})|dxdy$$
$$\le C\int_0^{+\infty}\int_{I\!R^n_+} \frac{y^A t^{B+1}}{(t^2+y^2)^\Gamma}\tilde{\phi}_i|\nabla\tilde{v}|dxdy + C\int_0^{+\infty}\int_{I\!R^n_+} \frac{y^A t^{B+1}}{(t^2+y^2)^\Gamma}|\nabla\tilde{\phi}_i||\tilde{v}|dxdy$$

Hence, returning to our original variables we have that

$$\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|\phi_i v|dxdy$$
$$\le C\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}\phi_i|\nabla v|dxdy + C\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|\nabla\phi_i||v|dxdy\ .$$

Summing over i we get that

$$\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy$$
$$\le C_1\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|\nabla v|dxdy + C_2\int_0^{+\infty}\int_{\Omega_\varepsilon} \frac{y^A d^{B+1}}{(d^2+y^2)^\Gamma}|v|dxdy\ .$$

The result then follows easily.

$\square$

When working in the complement of Ω we have the following surprising result:

Lemma 4.4. *Let $\Omega \subset I\!R^n$. If A, B, Γ are constants such that $A+1>0$, $B+1>0$ and $2\Gamma < A+B+2$ then for all $v \in C_0^\infty(I\!R^n \times I\!R)$ there holds*

$$(A+1)(A+B+2-2\Gamma^+)\int_0^{+\infty}\int_{\mathcal{C}\Omega} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy \le \tag{4.14}$$
$$2\Gamma^+\int_0^{+\infty}\int_{\mathcal{C}\Omega} \frac{y^{A+2} d^{B+1}}{(d^2+y^2)^{\Gamma+1}}(-\Delta d)|v|dxdy + (A+B+2)\int_0^{+\infty}\int_{\mathcal{C}\Omega} \frac{y^{A+1} d^B}{(d^2+y^2)^\Gamma}|\nabla v|dxdy\ ,$$

where $\Gamma^+ = \max(0,\Gamma)$.

We note that no assumption on the sign of $-\Delta d$ is required.

Proof: Integrating by parts in the y-variable we compute

$$(A+1)\int_0^{+\infty}\int_{\mathcal{C}\Omega} \frac{y^A d^B}{(d^2+y^2)^\Gamma}|v|dxdy \le 2\Gamma\int_0^{+\infty}\int_{\mathcal{C}\Omega} \frac{y^{A+2} d^B}{(d^2+y^2)^{\Gamma+1}}|v|dxdy$$
$$+\int_0^{+\infty}\int_{\mathcal{C}\Omega} \frac{y^{A+1} d^B}{(d^2+y^2)^\Gamma}|\nabla v|dxdy. \tag{4.15}$$

If $\Gamma \le 0$ the result follows easily. In the sequel we consider the case $\Gamma > 0$. In the previous calculation there is no boundary term due to our assumptions. To continue we will estimate the first term in the right hand side above. To this end we define the vector field $\vec{F}$ by

$$\vec{F}(x,y) := \left(\frac{y^{A+2}d^{B+3}\nabla d}{(d^2+y^2)^{\Gamma+1}}, \frac{y^{A+3}d^B}{(d^2+y^2)^{\Gamma+1}} \right). \tag{4.16}$$

We then have

$$\int_0^{+\infty}\int_{\mathcal{C}\Omega} \mathrm{div}\vec{F}|v|dxdy = -\int_0^{+\infty}\int_{\mathcal{C}\Omega} \vec{F}\cdot\nabla|v|dxdy \le \int_0^{+\infty}\int_{\mathcal{C}\Omega} |\vec{F}||\nabla v|dxdy. \tag{4.17}$$

We note that because of our assumptions $A+1>0$ and $B+1>0$, there are no boundary terms in (4.17). Straightforward calculations show that

$$\mathrm{div}\vec{F} = \frac{y^{A+2}d^{B+1}(\Delta d)}{(d^2+y^2)^{\Gamma+1}} + (A+B+2-2\Gamma)\frac{y^{A+2}d^B}{(d^2+y^2)^{\Gamma+1}}, \tag{4.18}$$

and

$$|\vec{F}| = \frac{y^{A+2}d^B}{(d^2+y^2)^{\Gamma+1/2}} \le \frac{y^{A+1}d^B}{(d^2+y^2)^{\Gamma}}. \tag{4.19}$$

Combining the above we conclude the proof. Again,we note that in all integrations by parts there are no boundary terms due to our assumptions.

□

As a consequence of Lemma 4.1 we have:

Lemma 4.5. *Let $\Omega \subset I\!R^n$ be such that $-\Delta d(x) \ge 0$, for $x \in \Omega$ and $w \in C_0^1(I\!R^n \times I\!R)$. If A, B, Γ are constants such that $A+1>0$, $B+1>0$, and $2\Gamma < A+B+2$, then,*

$$\frac{(B+1)^2(B+A+2-2\Gamma^+)^2}{4(B+A+2)^2}\int_0^{+\infty}\int_\Omega \frac{y^Ad^B}{(d^2+y^2)^\Gamma}w^2dxdy \quad \le \tag{4.20}$$

$$\frac{(B+1)(B+A+2-2\Gamma^+)}{2(B+A+2)}\int_0^{+\infty}\int_\Omega \frac{y^Ad^{B+1}}{(d^2+y^2)^\Gamma}(-\Delta d)w^2dxdy \quad + \int_0^{+\infty}\int_\Omega \frac{y^Ad^{B+2}}{(d^2+y^2)^\Gamma}|\nabla w|^2dxdy\ ,$$

where $\Gamma^+ = \max(0,\Gamma)$.

Proof: We apply Lemma 4.1 to $v = w^2$. To conclude we use Young's inequality in the last term of the right hand side. We omit the details.

□

In the case where $A+B+2 = 2\Gamma$ the L^2 analogue of Lemma 4.2 reads:

Lemma 4.6. *Suppose that $\Omega \subset I\!R^n$ has finite inner radius and is such that $-\Delta d(x) \ge 0$ for $x \in \Omega$. If A, B are constants such that $A+1>0$, $B+1>0$, then for all $w \in C_0^\infty(I\!R^n \times I\!R)$ there holds*

$$\left(\frac{B+1}{2(A+B+3)}\right)^2\int_0^{+\infty}\int_\Omega \frac{y^Ad^BX^2}{(d^2+y^2)^{\frac{A+B+2}{2}}}w^2dxdy \quad \le \tag{4.21}$$

$$\frac{B+1}{2(A+B+3)}\int_0^{+\infty}\int_\Omega \frac{y^Ad^{B+1}X}{(d^2+y^2)^{\frac{A+B+2}{2}}}(-\Delta d)w^2dxdy \quad + \int_0^{+\infty}\int_\Omega \frac{y^Ad^{B+2}}{(d^2+y^2)^{\frac{A+B+2}{2}}}|\nabla w|^2dxdy\ ,$$

where $X = X(\frac{d(x)}{R_{in}})$ and $X(t) = (1-\ln t)^{-1}$, $0 < t \le 1$.

Proof: We apply Lemma 4.2 to $v = w^2$. To conclude we use Young's inequality in the last term of the right hand side. We omit the details.

□

In the case of half space a more delicate result is needed. More precisely we have:

Lemma 4.7. *Let $v \in C_0^\infty(I\!R^n \times I\!R)$. If $0 < A \leq \frac{1}{2}$, $B+1>0$, and $2\Gamma < A+B+2$, then the following inequality holds true:*

$$c_0 \int_0^{+\infty} \int_{I\!R^n_+} \frac{y^{-A} x_n^B}{(x_n^2+y^2)^{\Gamma-A}} |v| dx dy \leq \int_0^{+\infty} \int_{I\!R^n_+} \frac{y^A x_n^{1+B}}{(x_n^2+y^2)^{\Gamma}} |\nabla v| dx dy \, , \tag{4.22}$$

where

$$c_0 = \frac{A(B+1)(B+A+2-2\Gamma^+)}{(A+B+2)(A+2B+2)-2\Gamma^+(B+1)} \, .$$

The same result holds true if we replace $I\!R^n_+$ by $I\!R^n_-$ with $|x_n|$ in the place of x_n.

Proof: We will use polar coordinates, $x_n = r\cos\theta$, $y = r\sin\theta$. We first establish the following inequality for the angular derivative.

$$\begin{aligned} A\int_0^{\frac{\pi}{2}} (\sin\theta)^{-A}(\cos\theta)^B |v| d\theta \quad &\leq \quad (1+A+B)\int_0^{\frac{\pi}{2}} (\sin\theta)^{1+A}(\cos\theta)^B |v| d\theta \\ &+ \quad \int_0^{\frac{\pi}{2}} (\sin\theta)^A (\cos\theta)^{1+B} |v_\theta| d\theta \, . \end{aligned} \tag{4.23}$$

We have

$$\begin{aligned} \frac{d}{d\theta}((\sin\theta)^A(\cos\theta)^{1+B}) \quad &= \quad A(\sin\theta)^{A-1}(\cos\theta)^{2+B} - (1+B)(\sin\theta)^{A+1}(\cos\theta)^B \\ &= \quad A(\sin\theta)^{A-1}(\cos\theta)^{B} - (1+A+B)(\sin\theta)^{A+1}(\cos\theta)^B \, , \end{aligned}$$

therefore an integration by parts gives:

$$\begin{aligned} A\int_0^{\frac{\pi}{2}} (\sin\theta)^{A-1}(\cos\theta)^B |v| d\theta \quad &\leq \quad (1+A+B)\int_0^{\frac{\pi}{2}} (\sin\theta)^{1+A}(\cos\theta)^B |v| d\theta \\ &+ \quad \int_0^{\frac{\pi}{2}} (\sin\theta)^A (\cos\theta)^{1+B} |v_\theta| d\theta \, . \end{aligned}$$

Since $A \leq \frac{1}{2}$ we also have that $(\sin\theta)^{-A} \leq (\sin\theta)^{A-1}$ and (4.23) follows.

We next multiply (4.23) by $r^{A+B+1-2\Gamma}$ and then integrate over $(0,\infty)$ to conclude:

$$\begin{aligned} A\int_0^{+\infty}\int_0^{+\infty} \frac{y^{-A}x_n^B}{(x_n^2+y^2)^{\Gamma-A}} |v| dx_n dy \quad &\leq \quad (1+A+B)\int_0^{+\infty}\int_0^{+\infty} \frac{y^{1+A}x_n^B}{(x_n^2+y^2)^{\Gamma+\frac{1}{2}}} |v| dx_n dy \\ &+ \quad \int_0^{+\infty}\int_0^{+\infty} \frac{y^A x_n^{1+B}}{(x_n^2+y^2)^{\Gamma}} |\nabla v| dx_n dy \\ &\leq \quad (1+A+B)\int_0^{+\infty}\int_0^{+\infty} \frac{y^A x_n^B}{(x_n^2+y^2)^{\Gamma}} |v| dx_n dy \\ &+ \quad \int_0^{+\infty}\int_0^{+\infty} \frac{y^A x_n^{1+B}}{(x_n^2+y^2)^{\Gamma}} |\nabla v| dx_n dy \, . \end{aligned} \tag{4.24}$$

We next estimate the first term in the right hand side by using Lemma 4.1, that is,

$$\frac{(B+1)(B+A+2-2\Gamma^+)}{B+A+2}\int_0^{+\infty}\int_0^{+\infty} \frac{y^A x_n^B}{(x_n^2+y^2)^{\Gamma}} |v| dx_n dy \leq \int_0^{+\infty}\int_0^{+\infty} \frac{y^A x_n^{B+1}}{(x_n^2+y^2)^{\Gamma}} |\nabla v| dx_n dy \, .$$

A further integration in the other variables completes the proof.

□

5 Half Space, Trace Hardy & Trace Hardy–Sobolev–Maz'ya Inequalities

Here we will prove the trace Hardy and trace Hardy–Sobolev–Maz'ya inequalities appearing in Theorems 1.2 and 1.6. We start with the trace Hardy inequalities.

5.1 Half Space, Trace Hardy I & II

In this subsection we will provide the proof of the trace Hardy inequalities appearing in Theorems 1.2 and 1.6.

Proof of Theorem 1.2 part (i) and (ii): The case where $s \in [\frac{1}{2}, 1)$ is contained in Theorem 1.1. We next consider the case $s \in (0, \frac{1}{2})$ or equivalently $a \in (0, 1)$.

We will use the notation $x = (x', x_n) \in I\!R^n_+$ with $x_n > 0$. We will use Lemma 2.1 with the test function ϕ given by

$$\phi(x, y) = x_n^{-\frac{a}{2}} A\left(\frac{y}{x_n}\right), \qquad y > 0, \quad x_n > 0, x \in I\!R^n_+ \,,$$

where A solves (2.2), (2.3). Using Proposition 2.2 and Lemma 2.3 we see that all hypotheses of Lemma 2.1 are satisfied. In particular, for $t = \frac{y}{x_n}$ we compute, for $x \in I\!R^n_+$,

$$-\lim_{y \to 0^+} \left(y^a \frac{\phi_y(x, y)}{\phi(x, y)} \right) = \frac{\bar{d}_s}{x_n^{1-a}} \,.$$

We also have

$$-\mathrm{div}(y^a \nabla \phi) = 0, \quad y > 0, \quad x \in I\!R^n_+ \,.$$

From Lemma 2.1 we get

$$\int_0^{+\infty} \int_{I\!R^n_+} y^a |\nabla u|^2 dx dy \geq \bar{d}_s \int_{I\!R^n_+} \frac{u^2(x, 0)}{x_n^{1-a}} dx + \int_0^{+\infty} \int_{I\!R^n_+} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dx dy \tag{5.1}$$

from which the trace Hardy inequality follows directly. This relation will be used later on, to obtain the Sobolev term as well.

The optimality of $\bar{d}_s$ follows by the same test functions given by (2.13) as in the flat case of Theorem 1.1. The fact that a covers the full interval $(-1, 1)$ does not affect the calculations leading to (2.16).

□

Proof of Theorem 1.6 part (i): The case where $s \in [\frac{1}{2}, 1)$ is contained in Theorem 1.4. We next consider the case $s \in (0, \frac{1}{2})$ or equivalently $a \in (0, 1)$. We will use Lemma 3.1 with the test function ϕ given

$$\phi(x, y) = (y^2 + x_n^2)^{-\frac{a}{4}} B(\frac{x_n}{y}), \qquad y > 0, \;\; x_n \in I\!R \,.$$

Using Proposition 3.2 and Lemma 3.3 we see that all hypotheses of Lemma 3.1 are satisfied. In particular we compute

$$-\lim_{y \to 0^+} \left(y^a \frac{\phi_y(x, y)}{\phi(x, y)} \right) = \frac{\bar{k}_s}{x_n^{1-a}} \,, \qquad x_n > 0 \,.$$

An easy calculation shows that

$$-\mathrm{div}(y^a \nabla \phi) = 0, \qquad x \in I\!R^n, \quad y > 0 \,.$$

We now use Lemma 3.1 to get

$$\int_0^{+\infty} \int_{I\!R^n} y^a |\nabla u|^2 dx dy \geq \bar{k}_s \int_{I\!R^n_+} \frac{u^2(x, 0)}{x_n^{1-a}} dx + \int_0^{+\infty} \int_{I\!R^n} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dx dy \tag{5.2}$$

from which the trace Hardy inequality follows directly. This relation will also be used later on, to obtain the Sobolev term as well.

The optimality of $\bar{k}_s$ follows by the same test functions given by (3.20) as in the flat case of Theorem 1.4. The fact that a covers the full interval $(-1, 1)$ does not affect the calculations leading to (3.22).

□

5.2 Half Space, Trace Hardy–Sobolev–Maz'ya I & II

Here we will give the proof of the trace Hardy–Sobolev–Maz'ya inequalities of Theorems 1.2 and 1.6. We will first establish different trace Hardy–Sobolev–Maz'ya inequalities where only the Hardy term appears in the trace, and which are of independent interest.

Theorem 5.1. *Let $0 < s < 1$ and $n \geq 2$. There exists a positive constant c such that for all $u \in C_0^\infty(I\!\!R^n_+ \times I\!\!R)$ there holds*

$$\int_0^{+\infty} \int_{I\!\!R^n_+} y^{1-2s} |\nabla_{(x,y)} u(x,y)|^2 dxdy \geq \bar{d}_s \int_{I\!\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx + c \left(\int_0^{+\infty} \int_{I\!\!R^n_+} |u(x,y)|^{\frac{2(n+1)}{n-2s}} dxdy \right)^{\frac{n-2s}{n+1}} . \tag{5.3}$$

with

$$\bar{d}_s := \frac{2\Gamma\left(1-s\right)\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)\Gamma\left(s\right)} . \tag{5.4}$$

Proof of Theorem 5.1: From the proof of Theorem 1.2 we recall the inequality (5.1), that is

$$\int_0^{+\infty} \int_{I\!\!R^n_+} y^a |\nabla u|^2 dxdy \geq \bar{d}_s \int_{I\!\!R^n_+} \frac{u^2(x,0)}{x_n^{1-a}} dx + \int_0^{+\infty} \int_{I\!\!R^n_+} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dxdy , \tag{5.5}$$

where ϕ is given by

$$\phi(x,y) = x_n^{-\frac{a}{2}} A\left(\frac{y}{x_n}\right), \qquad y > 0, \quad x_n > 0, \quad x \in I\!\!R^n_+ ,$$

and A solves (2.2), (2.3).

The result will follow after establishing the following inequality:

$$\int_0^{+\infty} \int_{I\!\!R^n_+} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dxdy \geq c \left(\int_0^{+\infty} \int_{I\!\!R^n_+} |u|^{\frac{2(n+1)}{n+a-1}} dxdy \right)^{\frac{n+a-1}{(n+1)}} . \tag{5.6}$$

To this end we start with the inequality, see [M], Theorem 1, section 2.1.6,

$$\int_0^{+\infty} \int_{I\!\!R^n_+} y^{\frac{a}{2}} |\nabla u| dxdy \geq c \left(\int_0^{+\infty} \int_{I\!\!R^n_+} |u(x,y)|^{\frac{2(n+1)}{2n+a}} dxdy \right)^{\frac{2n+a}{2(n+1)}} , \qquad u \in C_0^\infty(I\!\!R^n_+ \times I\!\!R) ,$$

with the choice $u = \phi^{\frac{2n+a}{n+a-1}} v$. Hence we obtain

$$\int_0^{+\infty} \int_{I\!\!R^n_+} y^{\frac{a}{2}} \phi^{\frac{2n+a}{n+a-1}} |\nabla v| dxdy + \frac{2n+a}{n+a-1} \int_0^{+\infty} \int_{I\!\!R^n_+} y^{\frac{a}{2}} \phi^{\frac{n+1}{n+a-1}} |\nabla \phi||v| dxdy$$
$$\geq c \left(\int_0^{+\infty} \int_{I\!\!R^n_+} |\phi^{\frac{2n+a}{n+a-1}} v|^{\frac{2(n+1)}{2n+a}} dxdy \right)^{\frac{2n+a}{2(n+1)}} . \tag{5.7}$$

Next we will control the second term of the LHS by the first term of the LHS. To this end we consider two cases. Suppose first that $s \in [\frac{1}{2}, 1)$ that is $a \in (-1, 0]$. Using the asymptotics of Lemma 2.3 we get that

$$y^{\frac{a}{2}} \phi^{\frac{n+1}{n+a-1}} |\nabla \phi| \sim \frac{y^{\frac{a}{2}} x_n^{\frac{n+1}{n+a-1}}}{(x_n^2 + y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}} ,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}} \sim \frac{y^{\frac{a}{2}}x_n^{\frac{2n+a}{n+a-1}}}{(x_n^2+y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}} \ . \tag{5.8}$$

The sought for estimate then is a consequence of Lemma 4.1 with the choice: $A=\frac{a}{2}$, $B=\frac{n+1}{n+a-1}$ and $\Gamma=\frac{(2+a)(2n+a)}{4(n+a-1)}$ taking into account that

$$A+B+2-2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0 \ .$$

We next consider the case $a\in(0,1)$. Using again the asymptotics of Lemma 2.3 this time we have that

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi| \sim \frac{y^{-\frac{a}{2}}x_n^{\frac{n+1}{n+a-1}}}{(x_n^2+y^2)^{\frac{(2+a)(n+1)}{4(n+a-1)}+\frac{2-a}{4}}} \ ,$$

whereas, (5.8) remains the same. The sought for estimate now is a consequence of Lemma 4.7 with the choice $A=\frac{a}{2}$, $B=\frac{n+1}{n+a-1}$ and $\Gamma=\frac{(2+a)(2n+a)}{4(n+a-1)}$ taking into account that

$$A+B+2-2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0 \ .$$

Therefore for any $a\in(-1,1)$ we arrive at:

$$\int_0^{+\infty}\int_{\mathbb{R}^n_+} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v| \geq c\left(\int_0^{+\infty}\int_{\mathbb{R}^n_+}|\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}} \ . \tag{5.9}$$

To continue we next set in (5.9) $v=|w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS to conclude after a simplification

$$\int_0^{+\infty}\int_{\mathbb{R}^n_+} y^a\phi^2|\nabla w|^2dxdy \geq c\left(\int_0^{+\infty}\int_{\mathbb{R}^n_+}|\phi w|^{\frac{2(n+1)}{n+a-1}}dxdy\right)^{\frac{n+a-1}{n+1}} \ , \tag{5.10}$$

which is equivalent to (5.6).

□

Proof of Theorem 1.2 part (iii): Our starting point now is the following weighted trace Sobolev inequality, see [M], Theorem 1, section 2.1.6,

$$\int_0^{+\infty}\int_{\mathbb{R}^n_+} y^{\frac{a}{2}}|\nabla u|dxdy \geq c\left(\int_{\mathbb{R}^n_+}|u(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}} \ , \qquad u\in C_0^\infty(\mathbb{R}^n_+\times\mathbb{R}) \ .$$

Again we set $u=\phi^{\frac{2n+a}{n+a-1}}v$, to obtain the analogue of (5.7).

$$\int_0^{+\infty}\int_{\mathbb{R}^n_+} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy + \frac{2n+a}{n+a-1}\int_0^{+\infty}\int_{\mathbb{R}^n_+} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy$$
$$\geq c\left(\int_{\mathbb{R}^n_+}|\phi^{\frac{2n+a}{n+a-1}}(x,0)v(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}} \ . \tag{5.11}$$

As in the proof of Theorem 5.1 we control the second term of the LHS by the first term of the LHS to arrive at

$$\int_0^{+\infty}\int_{\mathbb{R}^n_+} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\left(\int_{\mathbb{R}^n_+}|\phi^{\frac{2n+a}{n+a-1}}(x,0)v(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}} \ .$$

Again, we set $v = |w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS to arrive at

$$\left(\int_0^{+\infty} \int_{I\!R^n_+} y^a \phi^2 |\nabla w|^2 dxdy \right)^{\frac{1}{2}} \left(\int_0^{+\infty} \int_{I\!R^n_+} |\phi w|^{\frac{2(n+1)}{n+a-1}} dxdy \right)^{\frac{1}{2}} \geq c \left(\int_{I\!R^n_+} |(\phi w)(x,0)|^{\frac{2n}{n+a-1}} dx \right)^{\frac{2n+a}{2n}} .$$

We next use (5.10) to conclude after a simplification

$$\int_0^{+\infty} \int_{I\!R^n_+} y^a \phi^2 |\nabla w|^2 dxdy \geq c \left(\int_{I\!R^n_+} |(\phi w)(x,0)|^{\frac{2n}{n+a-1}} dx \right)^{\frac{n+a-1}{n}} ,$$

which is equivalent to

$$\int_0^{+\infty} \int_{I\!R^n_+} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dxdy \geq c \left(\int_{I\!R^n_+} |u(x,0)|^{\frac{2n}{n+a-1}} dx \right)^{\frac{n+a-1}{n}} .$$

Combining this with inequality (5.1) we conclude the proof.

□

We next present a preliminary result which will play an important role towards establishing the Hardy–Sobolev–Maz'ya II of Theorem 1.6.

Theorem 5.2. *Let $0 < s < 1$ and $n \geq 2$. There exists a positive constant c, such that for all $u \in C_0^\infty(I\!R^n \times I\!R)$ with $u(x,0) = 0$, $x \in I\!R^n_-$, there holds*

$$\int_0^{+\infty} \int_{I\!R^n} y^{1-2s} |\nabla_{(x,y)} u(x,y)|^2 dxdy \geq \bar{k}_s \int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{2s}} dx + c \left(\int_0^{+\infty} \int_{I\!R^n} |u(x,y)|^{\frac{2(n+1)}{n-2s}} dxdy \right)^{\frac{n-2s}{n+1}} , \tag{5.12}$$

where

$$\bar{k}_s := \frac{2^{1-2s} \Gamma^2(s + \frac{1}{2}) \Gamma(1-s)}{\pi \Gamma(s)} ,$$

is the best constant in (5.12).

Proof: From the proof of Theorem 1.4 we recall the inequality (3.18), that is

$$\int_0^{+\infty} \int_{I\!R^n} y^a |\nabla u|^2 dxdy \geq \bar{k}_s \int_{I\!R^n_+} \frac{u^2(x,0)}{x_n^{1-a}} dx + \int_0^{+\infty} \int_{I\!R^n} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dxdy , \tag{5.13}$$

where ϕ is given by

$$\phi(x,y) = (y^2 + x_n^2)^{-\frac{a}{4}} B(\frac{x_n}{y}), \qquad y > 0, \ \ x_n \in I\!R ,$$

and B solves (3.3), (3.4).

Again, the result will follow after establishing the following inequality:

$$\int_0^{+\infty} \int_{I\!R^n} y^a |\nabla u - \frac{\nabla \phi}{\phi} u|^2 dxdy \geq c \left(\int_0^{+\infty} \int_{I\!R^n} |u|^{\frac{2(n+1)}{n+a-1}} dxdy \right)^{\frac{n+a-1}{n+1}} . \tag{5.14}$$

To this end we start with the inequality, see [M], Theorem 1, section 2.1.6,

$$\int_0^{+\infty} \int_{I\!R^n} y^{\frac{a}{2}} |\nabla u| dxdy \geq c \left(\int_0^{+\infty} \int_{I\!R^n} |u(x,y)|^{\frac{2(n+1)}{2n+a}} dxdy \right)^{\frac{2n+a}{2(n+1)}} , \qquad u \in C_0^\infty(I\!R^n \times I\!R) ,$$

with the choice $u = \phi^{\frac{2n+a}{n+a-1}} v$. Hence we obtain

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy + \frac{2n+a}{n+a-1}\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy$$
$$\geq c\left(\int_0^{+\infty}\int_{I\!R^n} |\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}} . \tag{5.15}$$

Next we will control the second term of the LHS by the first term of the LHS. To this end we consider various cases. Suppose first that $s \in [\frac{1}{2}, 1)$ that is $a \in (-1, 0]$ and $x \in I\!R^n_+$. Using the asymptotics of Lemma 3.3 we get that

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi| \sim \frac{y^{\frac{a}{2}}}{(x_n^2+y^2)^{\frac{a(n+1)}{4(n+a-1)}+\frac{a+2}{4}}} ,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}} \sim \frac{y^{\frac{a}{2}}}{(x_n^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}}} . \tag{5.16}$$

We now apply Lemma 4.1 with the choice: $A = \frac{a}{2}$, $B = 0$ and $\Gamma = \frac{a(n+1)}{4(n+a-1)} + \frac{a+2}{4}$ taking into account that

$$A + B + 2 - 2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0 .$$

Thus we get for some positive constant c that

$$\int_0^{+\infty}\int_{I\!R^n_+} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\int_0^{+\infty}\int_{I\!R^n_+} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy . \tag{5.17}$$

We next consider the case $a \in (0, 1)$, $x \in I\!R^n_+$. In this case

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi| \sim \frac{y^{-\frac{a}{2}}}{(x_n^2+y^2)^{\frac{a(n+1)}{4(n+a-1)}+\frac{2-a}{4}}} ,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}} \sim \frac{y^{\frac{a}{2}}}{(x_n^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}}} . \tag{5.18}$$

We now use Lemma 4.7 with the choice $A = \frac{a}{2}$, $B = 0$ and $\Gamma = \frac{1}{2} + \frac{a(2n+a)}{4(n+a-1)}$ taking into account that $\frac{x_n}{(x_n^2+y^2)^{\frac{1}{2}}} < 1$ and $A + B + 2 - 2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0$. We then conclude that (5.17) is valid for all $a \in (-1, 1)$.

In a similar manner for all $a \in (-1, 1)$ and $x \in I\!R^n_-$ we get that

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi| \sim \frac{y^{-\frac{a}{2}+\frac{(1-a)(n+1)}{n+a-1}}}{(x_n^2+y^2)^{\frac{(2-a)(2n+a)}{4(n+a-1)}}} ,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}} \sim \frac{y^{\frac{a}{2}+\frac{(1-a)(2n+a)}{n+a-1}}}{(x_n^2+y^2)^{\frac{(2-a)(2n+a)}{4(n+a-1)}}} . \tag{5.19}$$

This time we use Lemma 4.4 with $A = -\frac{a}{2} + \frac{(1-a)(n+1)}{n+a-1}$, $B = 0$ and $\Gamma = \frac{(2-a)(2n+a)}{4(n+a-1)}$, noticing that

$$A + B + 2 - 2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0 ,$$

thus obtaining

$$\int_0^{+\infty}\int_{I\!R^n_-} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\int_0^{+\infty}\int_{I\!R^n_-} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy\;. \tag{5.20}$$

Combining (5.17) and (5.20) we obtain the following L^1 Hardy estimate on the whole $I\!R^n$:

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy\;. \tag{5.21}$$

Using this in (5.15) we get that

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\left(\int_0^{+\infty}\int_{I\!R^n} |\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}}\;. \tag{5.22}$$

To continue we next set in (5.22) $v=|w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS to conclude after a simplification

$$\int_0^{+\infty}\int_{I\!R^n} y^{a}\phi^{2}|\nabla w|^2dxdy \geq c\left(\int_0^{+\infty}\int_{I\!R^n} |\phi w|^{\frac{2(n+1)}{n+a-1}}dxdy\right)^{\frac{n+a-1}{n+1}}\;, \tag{5.23}$$

which is equivalent to (5.14). The result then follows.

$\square$

We are now ready to establish the Proof of Theorem 1.6 part (ii).

Proof of Theorem 1.6 part (ii): Again we will use inequality (5.13). This time the result will follow once we will establish the following inequality:

$$\int_0^{+\infty}\int_{I\!R^n} y^{a}|\nabla u-\frac{\nabla\phi}{\phi}u|^2dxdy \geq c\left(\int_{I\!R^n_+} |u(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{n+a-1}{n}}\;, \tag{5.24}$$

with ϕ given by

$$\phi(x,y)=(y^2+x_n^2)^{-\frac{a}{4}}B(\frac{x_n}{y}), \qquad y>0,\;\; x_n\in I\!R\;,$$

and B solves (3.3), (3.4).

Our starting point is again the following weighted trace Sobolev inequality, see [M], Theorem 1, section 2.1.6, valid for functions $u\in C_0^\infty(I\!R^n\times I\!R)$ with $u(x,0)=0$, $x\in I\!R^n_-$:

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}|\nabla u|dxdy \geq c\left(\int_{I\!R^n_+} |u(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}}\;.$$

We set $u=\phi^{\frac{2n+a}{n+a-1}}v$ to obtain

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy+\frac{2n+a}{n+a-1}\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy$$
$$\geq c\left(\int_{I\!R^n_+} |\phi^{\frac{2n+a}{n+a-1}}(x,0)v(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}}\;. \tag{5.25}$$

Combining this with (5.21) we get that

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\left(\int_{I\!R^n_+} |\phi^{\frac{2n+a}{n+a-1}}(x,0)v(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}}\;. \tag{5.26}$$

We set $v = |w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS to arrive at

$$\left(\int_0^{+\infty}\int_{I\!R^n} y^a\phi^2|\nabla w|^2dxdy\right)^{\frac{1}{2}}\left(\int_0^{\infty}\int_{I\!R^n}|\phi w|^{\frac{2(n+1)}{n+a-1}}dxdy\right)^{\frac{1}{2}} \geq c\left(\int_{I\!R^n_+}|(\phi w)(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{2n+a}{2n}} .$$

We next use the Sobolev inequality (5.23) to conclude after a simplification

$$\int_0^{+\infty}\int_{I\!R^n} y^a\phi^2|\nabla w|^2dxdy \geq c\left(\int_{I\!R^n_+}|(\phi w)(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{n+a-1}{n}} ,$$

which is equivalent to (5.24) and the result follows.

□

6 The General Case, Trace Hardy–Sobolev–Maz'ya I & II

6.1 Trace Hardy–Sobolev–Maz'ya I

Here we will give the proof of Theorem 1.1 part (iii). We first establish the following Hardy–Sobolev–Maz'ya where only the Hardy term appears in the trace term.

Theorem 6.1. *Let $\frac{1}{2} < s < 1$, $n \geq 2$ and $\Omega \subsetneqq I\!R^n$ be a uniformly Lipschitz domain with finite inner radius that in addition satisfies*

$$-\Delta d(x) \geq 0, \qquad x \in \Omega . \tag{6.1}$$

Then there exists a positive constant c such that for all $u \in C_0^\infty(\Omega \times I\!R)$ there holds

$$\int_0^{+\infty}\int_\Omega y^{1-2s}|\nabla_{(x,y)}u(x,y)|^2dxdy \geq \bar{d}_s\int_\Omega \frac{u^2(x,0)}{d^{2s}(x)}dx + c\left(\int_0^{+\infty}\int_\Omega |u(x,y)|^{\frac{2(n+1)}{n-2s}}dxdy\right)^{\frac{n-2s}{n+1}} . \tag{6.2}$$

with

$$\bar{d}_s := \frac{2\Gamma(1-s)\Gamma^2\left(\frac{3+2s}{4}\right)}{\Gamma^2\left(\frac{3-2s}{4}\right)\Gamma(s)} . \tag{6.3}$$

Proof of Theorem 6.1: From the proof of Theorem 1.1 we recall the inequality (2.11), that is

$$\begin{aligned}\int_0^{+\infty}\int_\Omega y^a|\nabla u|^2dxdy \geq \quad & \bar{d}_s\int_\Omega \frac{u^2(x,0)}{d^{1-a}(x)}dx + \int_0^{+\infty}\int_\Omega y^a|\nabla u - \frac{\nabla\phi}{\phi}u|^2dxdy \\ & - \int_0^{+\infty}\int_\Omega \frac{\operatorname{div}(y^a\nabla\phi)}{\phi}u^2dxdy ,\end{aligned} \tag{6.4}$$

where ϕ is given by

$$\phi(x,y) = d^{-\frac{a}{2}}(x)A\left(\frac{y}{d}\right), \qquad y > 0, \quad x \in \Omega , \tag{6.5}$$

and A solves (2.2), (2.3).

The result will follow after establishing the following inequality:

$$\int_0^{+\infty}\int_\Omega y^a|\nabla u - \frac{\nabla\phi}{\phi}u|^2dxdy - \int_0^{+\infty}\int_\Omega \frac{\operatorname{div}(y^a\nabla\phi)}{\phi}u^2dxdy \geq c\left(\int_0^{+\infty}\int_\Omega |u(x,y)|^{\frac{2(n+1)}{n+a-1}}dxdy\right)^{\frac{n+a-1}{n+1}} . \tag{6.6}$$

To this end we start with the inequality, see [M], Theorem 1, section 2.1.6,

$$\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}|\nabla u|dxdy \geq c\left(\int_0^{+\infty}\int_\Omega |u(x,y)|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}} , \qquad u \in C_0^\infty(\Omega \times I\!R) ,$$

with the choice $u=\phi^{\frac{2n+a}{n+a-1}}v$. Hence we obtain

$$\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy+\frac{2n+a}{n+a-1}\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy$$
$$\geq c\left(\int_0^{+\infty}\int_\Omega|\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}}\ . \tag{6.7}$$

Next we will control the second term of the LHS using Lemma 4.3. To this end we recall that for $a\in(-1,0)$ we have the following asymptotics from Lemma 2.3:

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi|\sim\frac{y^{\frac{a}{2}}d^{\frac{n+1}{n+a-1}}}{(d^2+y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}}\ ,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}\sim\frac{y^{\frac{a}{2}}d^{\frac{2n+a}{n+a-1}}}{(d^2+y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}}\ . \tag{6.8}$$

We then use Lemma 4.3 with the choice $A=\frac{a}{2}$, $B=\frac{n+1}{n+a-1}$ and $\Gamma=\frac{(2+a)(2n+a)}{4(n+a-1)}$ taking into account that

$$A+B+2-2\Gamma=\frac{(2-a)(n-1)}{2(n+a-1)}>0\ ,$$

to obtain the estimate

$$\int_0^{+\infty}\int_\Omega\frac{y^{\frac{a}{2}}d^{\frac{n+1}{n+a-1}}}{(d^2+y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}}|v|dxdy\leq C_1\int_0^{+\infty}\int_\Omega\frac{y^{\frac{a}{2}}d^{\frac{2n+a}{n+a-1}}}{(d^2+y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}}|\nabla v|dxdy$$
$$+C_2\int_0^{+\infty}\int_\Omega\frac{y^{\frac{a}{2}}d^{\frac{2n+a}{n+a-1}}}{(d^2+y^2)^{\frac{(2+a)(2n+a)}{4(n+a-1)}}}|v|dxdy\ .$$

From this and (6.7) we have that

$$\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy+\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|v|dxdy\geq c\left(\int_0^{+\infty}\int_\Omega|\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}}$$

To continue we next set $v=|w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS. After a simplification we arrive at:

$$\int_0^{+\infty}\int_\Omega y^a\phi^2|\nabla w|^2dxdy+\int_0^{+\infty}\int_\Omega y^a\phi^2w^2dxdy\geq C\left(\int_0^{+\infty}\int_\Omega|\phi w|^{\frac{2(n+1)}{n+a-1}}\right)^{\frac{n+a-1}{n+1}} \tag{6.9}$$

To conclude the proof of the Theorem we need the following estimate:

$$c\int_0^{+\infty}\int_\Omega y^a\phi^2w^2dxdy\leq\int_0^{+\infty}\int_\Omega y^a\phi^2|\nabla w|^2dxdy-\int_0^{+\infty}\int_\Omega\mathrm{div}(y^a\nabla\phi)\phi w^2dxdy\ . \tag{6.10}$$

It is here that we will use the fact that the domain Ω has finite inner radius. Using Lemma 4.6 with $A=a$, $B=0$ we obtain that

$$c\int_0^{+\infty}\int_\Omega\frac{y^aX^2\left(\frac{d}{R_{in}}\right)}{(d^2+y^2)^{\frac{2+a}{2}}}w^2dxdy\leq\int_0^{+\infty}\int_\Omega\frac{y^ad^2}{(d^2+y^2)^{\frac{2+a}{2}}}|\nabla w|^2dxdy-\int_0^{+\infty}\int_\Omega\frac{y^ad(\Delta d)X\left(\frac{d}{R_{in}}\right)}{(d^2+y^2)^{\frac{2+a}{2}}}w^2dxdy\ ,$$

which implies

$$c\int_0^{+\infty}\int_\Omega \frac{y^a d^2}{(d^2+y^2)^{\frac{2+a}{2}}} w^2 dxdy \le \int_0^{+\infty}\int_\Omega \frac{y^a d^2}{(d^2+y^2)^{\frac{2+a}{2}}} |\nabla w|^2 dxdy - \int_0^{+\infty}\int_\Omega \frac{y^a d(\Delta d)}{(d^2+y^2)^{\frac{2+a}{2}}} w^2 dxdy\,.$$

Taking into account the asymptotics of ϕ this is equivalent to (6.10). We omit further details.

□

We are now ready to prove Theorem 1.1 part (iii).

Proof of Theorem 1.1 part (iii): Again we will use (6.4). The result then will follow once we establish:

$$\int_0^{+\infty}\int_\Omega y^a |\nabla u - \frac{\nabla\phi}{\phi}u|^2 dxdy - \int_0^{+\infty}\int_\Omega \frac{\mathrm{div}(y^a\nabla\phi)}{\phi}u^2 dxdy \ge c\left(\int_\Omega |u(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{n+a-1}{n}}\,. \tag{6.11}$$

where ϕ is as in (6.5). To this end we start with the inequality, see [M], Theorem 1, section 2.1.6,

$$\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}|\nabla u| dxdy \ge c\left(\int_\Omega |u(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}}\,, \qquad u\in C_0^\infty(\Omega\times I\!R)\,,$$

with the choice $u=\phi^{\frac{2n+a}{n+a-1}}v$. Hence we obtain

$$\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v| dxdy + \frac{2n+a}{n+a-1}\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v| dxdy$$
$$\ge c\left(\int_\Omega |\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}}\,. \tag{6.12}$$

Next we will control the second term of the LHS exactly as we did in the proof of Theorem 6.1, to arrive at

$$\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v| dxdy + \int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|v| dxdy \ge c\left(\int_\Omega |\phi^{\frac{2n+a}{n+a-1}}v(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}}\,.$$

To continue we next set $v=|w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS to get after elementary manipulations that

$$\left(\int_0^{+\infty}\int_\Omega |\phi w|^{\frac{2(n+1)}{n+a-1}}dxdy\right)\left[\int_0^{+\infty}\int_\Omega y^a\phi^2|\nabla w|^2 dxdy + \int_0^{+\infty}\int_\Omega y^a\phi^2 w^2 dxdy\right]$$
$$\ge C\left(\int_\Omega |\phi w(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{2n+a}{n}}\,. \tag{6.13}$$

At this point we use Theorem 6.1 and inequality (6.10) to conclude the result. We omit further details.

□

6.2 Trace Hardy–Sobolev–Maz'ya II

Here we will give the proof of Theorem 1.4 part (iii). We first establish the following Hardy–Sobolev–Maz'ya where only the Hardy term appears in the trace term.

Theorem 6.2. *Let $\frac{1}{2}<s<1$, $n\ge 2$ and $\Omega\subsetneq I\!R^n$ be a uniformly Lipschitz and convex domain with finite inner radius. Then, there exists a positive constant c such that for all $u\in C_0^\infty(I\!R^n\times I\!R)$ with $u(x,0)=0$ for $x\in\mathcal{C}\Omega$ there holds*

$$\int_0^{+\infty}\int_{I\!R^n} y^{1-2s}|\nabla_{(x,y)}u(x,y)|^2 dxdy \ge \bar{k}_s\int_\Omega \frac{u^2(x,0)}{d^{2s}(x)}dx + c\left(\int_0^{+\infty}\int_{I\!R^n}|u(x,y)|^{\frac{2(n+1)}{n-2s}}dxdy\right)^{\frac{n-2s}{n+1}}\,. \tag{6.14}$$

with

$$\bar{k}_s := \frac{2^{1-2s}\Gamma^2(s+\frac{1}{2})\Gamma(1-s)}{\pi\Gamma(s)}\,. \tag{6.15}$$

Proof of Theorem 6.2: From the proof of Theorem 1.4 we recall the inequality (3.18), that is

$$\int_0^{+\infty}\int_{I\!R^n} y^a|\nabla u|^2 dxdy \geq \bar{k}_s\int_\Omega \frac{u^2(x,0)}{d^{1-a}(x)}dx + \int_0^{+\infty}\int_{I\!R^n} y^a|\nabla u - \frac{\nabla\phi}{\phi}u|^2 dxdy - \int_0^{+\infty}\int_{I\!R^n} \frac{\operatorname{div}(y^a\nabla\phi)}{\phi}u^2 dxdy\ , \tag{6.16}$$

where ϕ is given by

$$\phi(x,y) = \begin{cases} (y^2+d^2)^{-\frac{a}{4}}B(\frac{d}{y}), & x\in\Omega,\quad y>0 \\ (y^2+d^2)^{-\frac{a}{4}}B(-\frac{d}{y}), & x\in\mathcal{C}\Omega,\ \ y>0\ , \end{cases} \tag{6.17}$$

and B is the solution of the boundary value problem (3.3) and (3.4). The result will follow after establishing the following inequality:

$$\int_0^{+\infty}\int_{I\!R^n} y^a|\nabla u - \frac{\nabla\phi}{\phi}u|^2 dxdy - \int_0^{+\infty}\int_{I\!R^n} \frac{\operatorname{div}(y^a\nabla\phi)}{\phi}u^2 dxdy \geq c\left(\int_0^{+\infty}\int_{I\!R^n} |u(x,y)|^{\frac{2(n+1)}{n+a-1}}dxdy\right)^{\frac{n+a-1}{n+1}}\ . \tag{6.18}$$

To this end we start with the inequality, see [M], Theorem 1, section 2.1.6,

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}|\nabla u|dxdy \geq c\left(\int_0^{+\infty}\int_{I\!R^n} |u(x,y)|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}}\ ,\qquad u\in C_0^\infty(I\!R^n\times I\!R)\ ,$$

with the choice $u=\phi^{\frac{2n+a}{n+a-1}}v$. Hence we obtain

$$\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy + \frac{2n+a}{n+a-1}\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy \geq c\left(\int_0^{+\infty}\int_{I\!R^n} |\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}}\ . \tag{6.19}$$

Again we want to control the second term of the LHS. This time we split the integral into the integral over Ω and the integral over $\mathcal{C}\Omega$. Concerning the integral over $\mathcal{C}\Omega$ we use the asymptotics of ϕ as given by Lemma 3.3 for $a\in(-1,0)$ to get that

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi| \sim \frac{y^{-\frac{a}{2}+\frac{(1-a)(n+1)}{n+a-1}}}{(d^2+y^2)^{\frac{(2-a)(2n+a)}{4(n+a-1)}}}\ ,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}} \sim \frac{y^{\frac{a}{2}+\frac{(1-a)(2n+a)}{n+a-1}}}{(d^2+y^2)^{\frac{(2-a)(2n+a)}{4(n+a-1)}}}\ .$$

This time we use Lemma 4.4 with $A=-\frac{a}{2}+\frac{(1-a)(n+1)}{n+a-1}$, $B=0$ and $\Gamma=\frac{(2-a)(2n+a)}{4(n+a-1)}$, noticing that

$$A+B+2-2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0\ ,$$

thus obtaining

$$\int_0^{+\infty}\int_{\mathcal{C}\Omega} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy \geq c\int_0^{+\infty}\int_{\mathcal{C}\Omega} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy\ , \tag{6.20}$$

where we also used the convexity of Ω.

On the other hand in Ω the asymptotics of ϕ are also given by Lemma 3.3 as follows:

$$y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi| \sim \frac{y^{\frac{a}{2}}}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}+\frac{1}{2}}} \,,$$

whereas,

$$y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}} \sim \frac{y^{\frac{a}{2}}}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}}} \,.$$

We next use Lemma 4.3 with the choice $A=\frac{a}{2}$, $B=0$ and $\Gamma=\frac{a(2n+a)}{4(n+a-1)}+\frac{1}{2}$ taking into account that

$$A+B+2-2\Gamma = \frac{(2-a)(n-1)}{2(n+a-1)} > 0 \,,$$

to obtain the estimate

$$\begin{aligned}
&\int_0^{+\infty}\int_\Omega \frac{y^{\frac{a}{2}}}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}+\frac{1}{2}}}|v|dxdy \\
\le\ & C_1\int_0^{+\infty}\int_\Omega \frac{y^{\frac{a}{2}}d}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}+\frac{1}{2}}}|\nabla v|dxdy + C_2\int_0^{+\infty}\int_\Omega \frac{y^{\frac{a}{2}}d}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}+\frac{1}{2}}}|v|dxdy \\
\le\ & C_1\int_0^{+\infty}\int_\Omega \frac{y^{\frac{a}{2}}}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}}}|\nabla v|dxdy + C_2\int_0^{+\infty}\int_\Omega \frac{y^{\frac{a}{2}}d}{(d^2+y^2)^{\frac{a(2n+a)}{4(n+a-1)}+\frac{1}{2}}}|v|dxdy \,.
\end{aligned}$$

Equivalently, this can be written as

$$\begin{aligned}
&C\int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v|dxdy \\
\le\ & \int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy + \int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}\frac{d}{(d^2+y^2)^{\frac{1}{2}}}|v|dxdy \,. \qquad (6.21)
\end{aligned}$$

Using (6.20) and (6.21) in (6.19) we arrive at

$$\begin{aligned}
&\int_0^{+\infty}\int_{I\!R^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v|dxdy + \int_0^{+\infty}\int_\Omega y^{\frac{a}{2}}\frac{d}{(d^2+y^2)^{\frac{1}{2}}}\phi^{\frac{2n+a}{n+a-1}}|v|dxdy \\
&\qquad\qquad \ge c\left(\int_0^{+\infty}\int_{I\!R^n} |\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2(n+1)}{2n+a}}dxdy\right)^{\frac{2n+a}{2(n+1)}} \,. \qquad (6.22)
\end{aligned}$$

To continue we next set $v=|w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS. After a simplification we arrive at:

$$\int_0^{+\infty}\int_{I\!R^n} y^a\phi^2|\nabla w|^2dxdy + \int_0^{+\infty}\int_\Omega \frac{y^a d^2\phi^2}{d^2+y^2}w^2dxdy \ge c\left(\int_0^{+\infty}\int_{I\!R^n}|\phi w|^{\frac{2(n+1)}{n+a-1}}\right)^{\frac{n+a-1}{n+1}} \qquad (6.23)$$

To conclude the proof of the Theorem it is enough to obtain the following estimate:

$$c\int_0^{+\infty}\int_\Omega \frac{y^a d^2\phi^2}{d^2+y^2}w^2dxdy \le \int_0^{+\infty}\int_\Omega y^a\phi^2|\nabla w|^2dxdy - \int_0^{+\infty}\int_\Omega \mathrm{div}(y^a\nabla\phi)\phi w^2dxdy \,. \qquad (6.24)$$

It is here that we will use the fact that the domain Ω has finite inner radius. Using Lemma 4.6 with $A=a$, $B=0$ we obtain that

$$c\int_0^{+\infty}\int_\Omega \frac{y^aX^2\left(\frac{d}{R_{in}}\right)}{(d^2+y^2)^{\frac{2+a}{2}}}w^2dxdy \le \int_0^{+\infty}\int_\Omega \frac{y^ad^2}{(d^2+y^2)^{\frac{2+a}{2}}}|\nabla w|^2dxdy - \int_0^{+\infty}\int_\Omega \frac{y^ad(\Delta d)X\left(\frac{d}{R_{in}}\right)}{(d^2+y^2)^{\frac{2+a}{2}}}w^2dxdy \,,$$

which implies

$$c\int_0^{+\infty}\int_\Omega \frac{y^a d^2}{(d^2+y^2)^{\frac{2+a}{2}}} w^2 dxdy \le \int_0^{+\infty}\int_\Omega \frac{y^a}{(d^2+y^2)^{\frac{a}{2}}}|\nabla w|^2 dxdy - \int_0^{+\infty}\int_\Omega \frac{y^a(\Delta d)}{(d^2+y^2)^{\frac{1+a}{2}}} w^2 dxdy \,.$$

Taking into account the asymptotics of ϕ this is equivalent to (6.24). We omit further details.

□

We are now ready to prove Theorem 1.4 part (iii).

Proof of Theorem 1.4 part (iii): Again we will use (6.16). The result then will follow once we establish:

$$\int_0^{+\infty}\int_{\mathbb{R}^n} y^a|\nabla u - \frac{\nabla\phi}{\phi}u|^2 dxdy - \int_0^{+\infty}\int_{\mathbb{R}^n} \frac{\operatorname{div}(y^a\nabla\phi)}{\phi}u^2 dxdy \ge c\left(\int_\Omega |u(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{n+a-1}{n}} . \tag{6.25}$$

where ϕ is as in (6.17). To this end we start again with the inequality,

$$\int_0^{+\infty}\int_{\mathbb{R}^n} y^{\frac{a}{2}}|\nabla u| dxdy \ge c\left(\int_\Omega |u(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}} ,$$

valid for $u\in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$ with $u(x,0)=0,\ \ x\in\mathcal{C}\Omega$. We apply this to $u=\phi^{\frac{2n+a}{n+a-1}}v$. Hence we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v| dxdy + \frac{2n+a}{n+a-1}\int_0^{+\infty}\int_{\mathbb{R}^n} y^{\frac{a}{2}}\phi^{\frac{n+1}{n+a-1}}|\nabla\phi||v| dxdy$$
$$\ge c\left(\int_\Omega |\phi^{\frac{2n+a}{n+a-1}}v|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}} . \tag{6.26}$$

Next we will control the second term of the LHS exactly as we did in the proof of Theorem 6.2, to arrive at

$$\int_0^{+\infty}\int_{\mathbb{R}^n} y^{\frac{a}{2}}\phi^{\frac{2n+a}{n+a-1}}|\nabla v| dxdy + \int_0^{+\infty}\int_\Omega \frac{y^{\frac{a}{2}}d}{(d^2+y^2)^{\frac{1}{2}}}\phi^{\frac{2n+a}{n+a-1}}|v| dxdy \ge c\left(\int_\Omega |\phi^{\frac{2n+a}{n+a-1}}v(x,0)|^{\frac{2n}{2n+a}}dx\right)^{\frac{2n+a}{2n}} .$$

To continue we next set $v=|w|^{\frac{2n+a}{n+a-1}}$ and apply Schwartz inequality in the LHS to get after elementary manipulations that

$$\left(\int_0^{+\infty}\int_\Omega |\phi w|^{\frac{2(n+1)}{n+a-1}}dxdy\right)\left[\int_0^{+\infty}\int_\Omega y^a\phi^2|\nabla w|^2 dxdy + \int_0^{+\infty}\int_\Omega \frac{y^a d^2}{d^2+y^2}\phi^2 w^2 dxdy\right]$$
$$\ge C\left(\int_\Omega |\phi w(x,0)|^{\frac{2n}{n+a-1}}dx\right)^{\frac{2n+a}{n}} . \tag{6.27}$$

At this point we use Theorem 6.2 and inequality (6.24) to conclude the result. We omit further details.

□

7 The Fractional Laplacians

In this section we will apply the previous results to establish the proofs of Theorems 1.3, 1.5 as well as of part (iii) of Theorem 1.6.

Proof of Theorem 1.3: Part (i) and (iii) follow from part (i) and (iii) of Theorem 1.1 taking into account the relation between the energy of the extended problem and the corresponding one of the fractional Laplacian, see subsection 8.1 and in particular relation (8.5).

We next prove part (ii). We will use the optimality of the constant $\bar{d}_s$ of Theorem 1.1, that is for each $\varepsilon>0$ there exists a $u_\varepsilon\in C_0^\infty(\Omega\times\mathbb{R})$ such that

$$\bar{d}_s+\varepsilon \ge \frac{\int_0^{+\infty}\int_\Omega y^{1-2s}|\nabla u_\varepsilon|^2 dxdy}{\int_\Omega \frac{u_\varepsilon^2(x,0)}{d^{2s}(x)}dx} ,$$

and let $f_\varepsilon(x) = u_\varepsilon(x,0)$. We will show that for some positive constant c,

$$d_s + c\varepsilon \geq \frac{((-\Delta)^s f_\varepsilon, f_\varepsilon)_\Omega}{\int_\Omega \frac{f_\varepsilon^2(x)}{d^{2s}(x)} dx} \ . \tag{7.1}$$

To this end let $\hat{u}_\varepsilon$ be the solution to the extended problem

$$\begin{aligned} \operatorname{div}(y^{1-2s}\nabla \hat{u}_\varepsilon(x,y)) &= 0, \qquad \text{in} \quad \Omega \times (0,\infty)\ , \\ \hat{u}_\varepsilon(x,y) &= 0, \qquad x \in \partial\Omega \times (0,\infty)\ , \\ \hat{u}_\varepsilon(x,0) &= f_\varepsilon(x)\ . \end{aligned}$$

The solution $\hat{u}_\varepsilon$ minimizes the energy and therefore

$$\int_0^{+\infty} \int_\Omega y^{1-2s} |\nabla \hat{u}_\varepsilon|^2 dx dy \leq \int_0^{+\infty} \int_\Omega y^{1-2s} |\nabla u_\varepsilon|^2 dx dy \ .$$

On the other hand using (8.5) we have

$$\int_0^{+\infty} \int_\Omega y^{1-2s} |\nabla \hat{u}_\varepsilon|^2 dx dy = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)} ((-\Delta)^s f_\varepsilon, f_\varepsilon)_\Omega \ ,$$

and (7.1) follows easily with $c = \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}$.

□

We next give the proof of Theorem 1.5

Proof of Theorem 1.5: Part (i) and (iii) follow from part (i) and (iii) of Theorem 1.4 taking into account the relation between the energy of the extended problem and the corresponding one of the fractional Laplacian, see subsection 8.2 and in particular relations (8.7)–(8.8).

The proof of part (ii) is quite similar to the proof of part (ii) of Theorem 1.3, the only difference being that the extension problem is now on the whole $I\!R^n$. We omit the details.

□

Finally estimate (1.33) of part (iii) of Theorem 1.6 follows at once from part (ii) of Theorem 1.6 and (8.7). Concerning estimate (1.34), it follows from (1.33) taking into account that for $x \in I\!R^n_+$,

$$\int_{I\!R^n_-} \frac{d\xi}{|x-\xi|^{n+2s}} = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+2s}{2}\right)}{2s\Gamma\left(\frac{n+2s}{2}\right)} \frac{1}{x_n^{2s}} \ ,$$

see, e.g., [BBC].

8 Appendix

8.1 Spectral Fractional Laplacian

Let $\Omega \subset I\!R^n$ be a bounded domain, and let λ_i and ϕ_i be the Dirichlet eigenvalues and eigenfunctions of the Laplacian, i.e. $-\Delta\phi_i = \lambda_i\phi_i$ in Ω, with $\phi_i = 0$ on $\partial\Omega$, normalized so that $\int_\Omega \phi_i^2 dx = 1$. Then, for $f(x) = \sum c_i\phi_i(x)$ we define

$$(-\Delta)^s f = \sum_{i=1}^{\infty} c_i \lambda_i^s \phi_i, \qquad 0 < s < 1. \tag{8.1}$$

We also have

$$((-\Delta)^s f, f)_\Omega = \int_\Omega f\ (-\Delta)^s f dx = \sum_{i=1}^{\infty} c_i^2 \lambda_i^s. \tag{8.2}$$

To the function $f(x)$ we associate the "extended" function $u(x,y)$, $x \in \Omega$, $y > 0$, given by

$$u(x,y) = \sum_{i=1}^{+\infty} c_i \phi_i(x) T(y\sqrt{\lambda_i}),$$

where $T(t)$ is the energetic solution of the ODE:

$$(t^{1-2s}T'(t))' - t^{1-2s}T(t) = 0, \quad \text{or} \quad T'' + \frac{1-2s}{t}T' - T = 0, \quad t \geq 0. \tag{8.3}$$

The solution of this can be taken from [AS], Section 9.6 and is given by

$$T(t) = \frac{2^{1-s}}{\Gamma(s)} t^s K_s(t), \tag{8.4}$$

where $K_s(t)$ denotes the modified Bessel function of second kind. The constant factor is chosen in such a way that $T(0) = 1$. As a consequence we also have $u(x,0) = f(x)$.

An easy calculation shows that $\mathrm{div}(y^{1-2s}\nabla(\phi_i(x)T(y\sqrt{\lambda_i})) = 0$ from which it follows that $\mathrm{div}(y^{1-2s}\nabla u) = 0$. An integration by parts then shows that

$$\begin{aligned}
\int_0^{+\infty}\int_\Omega y^{1-2s}|\nabla u|^2 dxdy &= \lim_{\tau\to+\infty} \tau^{1-2s}\int_\Omega u(x,\tau)u_y(x,\tau)dx - \lim_{\tau\to 0}\tau^{1-2s}\int_\Omega u(x,\tau)u_y(x,\tau)dx \\
&= \left[\lim_{t\to+\infty} t^{1-2s}T(t)T'(t) - \lim_{t\to 0} t^{1-2s}T(t)T'(t)\right]\sum_{i=1}^{\infty}\lambda_i^s c_i^2 \\
&= \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}((-\Delta)^s f, f)_\Omega\ .
\end{aligned} \tag{8.5}$$

Where we used (8.2) and the fact that

$$\lim_{t\to+\infty} t^{1-2s}T(t)T'(t) - \lim_{t\to 0} t^{1-2s}T(t)T'(t) = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}\ . \tag{8.6}$$

To prove the above relation we show that

$$\lim_{t\to+\infty} t^{1-2s}T(t)T'(t) = 0, \qquad -\lim_{t\to 0} t^{1-2s}T(t)T'(t) = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}.$$

These two relations are a direct consequence of (8.4) and the following properties of $K_s(t)$:

$$\begin{aligned}
K_s(t) &\sim \frac{\Gamma(s)}{2^{1-s}}t^{-s}, \quad t\to 0, \qquad K_s(t) \sim \sqrt{\frac{\pi}{2t}}e^{-t}, \qquad t\to+\infty, \\
\frac{d}{dt}(t^sK_s(t)) &= -t^sK_{s-1}(t), \qquad\qquad K_s(t) = K_{-s}(t)\ .
\end{aligned}$$

8.2 Dirichlet Fractional Laplacian

Let $u(x,y)$ be the extended function as defined in (1.7)–(1.8). In this subsection we will show the following two relations connecting the energy of the extended problem and the energy of the Dirichlet fractional Laplacian:

$$\int_0^{+\infty}\int_{\mathbb{R}^n} y^{1-2s}|\nabla u|^2 dxdy = \frac{s\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(s)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2s}}dxd\xi \tag{8.7}$$

$$\int_0^{+\infty}\int_{\mathbb{R}^n} y^{1-2s}|\nabla u|^2 dxdy = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}((-\Delta)^s f, f)_{\mathbb{R}^n}\ . \tag{8.8}$$

We will use the Fourier transform in the x-variables:

$$\hat{u}(\eta, y) = (2\pi)^{-\frac{n}{2}} \int_{I\!R^n} e^{-ix\cdot\eta} u(x,y)dx.$$

The equation $\mathrm{div}(y^{1-2s}\nabla u(x,y)) = 0$ or equivalently $\Delta_x u + u_{yy} + \frac{a}{y}u_y = 0$ with $u(x,0) = f(x)$, reads as follows when taking the Fourier transform

$$-|\eta|^2\hat{u} + (\hat{u})_{yy} + \frac{1-2s}{y}(\hat{u})_y = 0, \qquad \hat{u}(\eta, 0) = \hat{f}(\eta),$$

and it is satisfied by $\hat{u}(\eta, y) = \hat{f}(\eta)T(|\eta|y)$, where T satisfies (8.3) and is given by (8.4).

Concerning the energies we have:

$$\begin{aligned}
\int_0^{+\infty} y^{1-2s} \int_{I\!R^n} |\nabla u|^2 dxdy &= \int_0^{+\infty} y^{1-2s} \int_{I\!R^n} \left(|\eta|^2|\hat{u}|^2 + |\hat{u}_y|^2\right) d\eta dy \\
&= \int_0^{+\infty} y^{1-2s} \int_{I\!R^n} \left\{|\eta|^2|\hat{f}|^2[T^2(|\eta|y) + T^{'2}(|\eta|y)]\right\} d\eta dy \\
&= \left(\int_{I\!R^n} |\eta|^{2s}|\hat{f}|^2 d\eta\right)\left(\int_0^\infty t^{1-2s}[T^2(t) + T^{'2}(t)]dt\right),
\end{aligned}$$

where $t = |\eta|y$. We next compute the last integral. Multiplying equation (8.3) by T, integrating by parts and employing (8.6), we get

$$\int_0^{+\infty} t^{1-2s}[T^2(t) + T^{'2}(t)]dt = t^{1-2s}T(t)T'(t)dt\Big|_0^\infty = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}. \tag{8.9}$$

We finally recall the following relation (see, e.g., [FLS], Lemma 3.1)

$$\begin{aligned}
\int_{I\!R^n} |\eta|^{2s}|\hat{f}|^2 d\eta &= \frac{c_{n,s}}{2} \int_{I\!R^n}\int_{I\!R^n} \frac{|f(x) - f(\xi)|^2}{|x-\xi|^{n+2s}} dxd\xi \\
&= \frac{s2^{2s-1}\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n}{2}}\Gamma(1-s)} \int_{I\!R^n}\int_{I\!R^n} \frac{|f(x) - f(\xi)|^2}{|x-\xi|^{n+2s}} dxd\xi.
\end{aligned} \tag{8.10}$$

Putting together the last three relations we conclude (8.7).

Finally, taking into account (1.25) we easily obtain (8.8).

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